Topology Preliminary Examination Spring 1992

Part I

1. Define the following: a topology, basis, open set, closed set, closure of a set, interior of a set, continuous function, Hausdorff space, regular space, normal space, and compact space.

2. Suppose $(X, \mathcal{T}_X)$ is a topological space and $A \subset X$.

   Prove or disprove:

   (a) $\overline{Int(A)} = \overline{A}$

   (b) $Int(Int(A)) = Int(A)$

   (c) If the set is finite, then $A = \overline{A}$.

3. Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, f is a continuous function from X onto Y, $A \subset X$, and $B \subset Y$.

   Prove or disprove:

   (a) If A is closed, then $f(A)$ is closed.

   (b) If B is closed, then $f^{-1}(B)$ is closed.

   (c) If A is connected, then $f(A)$ is connected.

   (d) If B is connected, then $f^{-1}(B)$ is connected.

Part II

Choose three problems from 4 to 10. The chosen problems should be worked completely; however, it is better to provide correct answers to only some of the subproblems than not quite correct answers to all of the subproblems.

4. Prove or disprove:

   (a) Every compact Hausdorff space is regular.

   (b) Every metric space is normal.

   (c) Each subspace of a separable Hausdorff space is separable.

5. Define homotopy quivalence, homotopy type, and covering space.

   (a) Prove that if $X = [0, 1]$ (with the usual topology) and $Y = \{0\}$ (one point), then X and Y are of the same homotopy type.

   (b) Describe, specifying the covering maps, all covering spaces of S^1.

6. State the Axiom of Choice, Zorn's Lemma, and the Well-ordering Principle.

   (a) Show that the Well-ordering Principle implies the Axiom of Choice.

   (b) Show another implication connecting the three axioms. That is, other than the implication in the previous subproblem, show that one of the three axioms implies one of the other two axioms.

   (c) Give an example of a proof (an easy proof is OK) that uses one of the three axioms. Now give an example of a proof that uses another one of the three axioms

7. Define a component and a path component.

   Prove or disprove:

   (a) Each component is a path component.

   (b) Each path component is a component.

   (c) A component is closed.

   (d) A path component is closed.

8. Define the space of all countable ordinals with the order topology.

   (a) What is the cardinality of the space of all countable ordinals?
   (b) Prove or disprove that the space of all countable ordinals is compact.
   (c) Prove or disprove that the space of all countable ordinals is countably compact.
   (d) Prove or disprove that the space of all countable ordinals is normal.

9. Define a retract and the projective plane.

   (a) Give, without proof, the first homology groups (with integral coefficients) of S^1, the projective plane, S^2, and $S^1 \times S^1$.
   (b) Prove, using homology, that S^1 is not a retract of S^2.

10. Define a paracompact space and Lindelöf space.

    Prove or disprove:

    (a) Every compact Hausdorff space is normal.
    (b) The real line, with the usual topology, is paracompact.
    (c) Every Hausdorff space is paracompact.