Topology Preliminary Exam

8/8/05

Solve eight out of the following problems, including **at least two out of problems 9-12**. Please do not use the same sheet of paper for two different problems. Use a cover sheet which lists the problems you have chosen to solve.

1. Prove that every compact subset of a Hausdorff space is closed.
2. Prove that if $\mathcal{H}$ is a collection of connected subsets of a topological space X such that $\cap\mathcal{H}$ is nonempty, then $\cup\mathcal{H}$ is a connected subset of X.
3. Prove that if X and Y are compact spaces, then $X \times Y$ is compact.
4. Prove that every metric space is normal.
5. Prove that the product of a countable collection of separable spaces is separable.
6. Suppose that $f : X \to Y$ is continuous and $H \subset X$ is connected. Show that $f(H)$ is connected.
7. Prove that the space X is compact if and only if every collection of closed subsets with the finite intersection property has nonempty intersection.
8. Let $\mathcal{C}(I, I)$ be the space of all continuous functions $f : I \to I$, where $I = [0, 1]$, and give $\mathcal{C}(I, I)$ the topology defined by the metric $d(f, g) = \sup\{|f(x) - g(x)| : 0 \leq x \leq 1\}$. Prove that $\mathcal{C}(I, I)$ is separable.
9. Let $X \subseteq Y$ be path-connected metric spaces with $x_0 \in X$, let $j : X \to Y$ be the inclusion map $j(x) = x$, and let $j_\star : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ be the induced homomorphism. Find such spaces X and Y so that

   (a) $j_\star$ is not 1–1.

   (b) $j_\star$ is not onto.

   (You may, if you wish, use the same example for both parts, but giving two different examples is also OK.)

10. Let E and B be path-connected metric spaces, and let n be a fixed positive integer. Let $p : E \to B$ be a continuous function such that $p^{-1}(b)$ has exactly n elements for each $b \in B$, and such that for every $e \in E$ there are neighborhoods U of e and V of $p(e)$ such that $p|U$ is a homeomorphism from U onto V. Prove that p is a covering map.

11. Let X be the square $[0,1] \times [0,1]$, and let Y be the quotient space obtained by identifying the four corners (i.e., $\{(0,0),(0,1),(1,1),(1,0)\}$ is one equivalence class and the remaining equivalence classes are singletons.) Prove that Y is not simply connected.

12. Prove that the fundamental group of the Klein Bottle is not Abelian.