Topology Prelim, August 18, 2010

1. Prove that X is Hausdorff iff every one point set equals the intersection of all of its closed neighborhoods.

2. Let $X = \mathbb{R}$. Define topology $\mathcal{T}$ on X as follows: Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}, n \neq 0\}$. For every $x \in X$ and $n \in \mathbb{N}$ let $U_n(x) = (x - \frac{1}{n}, x + \frac{1}{n})$ and

$$\mathcal{B}(x) = \begin{cases} \{U_n(x)\}_{n=1}^{\infty} & \text{if } x \neq 0 \\ \{U_n(x) - K\}_{n=1}^{\infty} & \text{if } x = 0 \end{cases}$$

   Show that $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}(x)$ is a basis.

3. Prove or disprove: The space defined in problem 2 is regular.

4. Define a compact space.
   (a) Prove that the Cartesian product (with product topology) of two compact spaces is compact.
   (b) Prove or disprove: If X is compact and $f : X \to Y$ is a continuous and onto function, then Y is compact.

5. Show that every metric space is normal.

6. Define the fundamental group.

7. Describe an example of a space with a non-abelian fundamental group. Explain why the group is non-abelian.

8. Define a covering space and describe all (up to the topological equivalence) connected covering spaces of the circle S^1.