Real Analysis Prelim   August 2013

No calculators, computers, phones, books, or notes. Do 4 problems on page 1, 4 problems on page 2, 2 problems on page 3. Problems on page 4 are bonus problems; you may do or not do any of them that you wish. Estimated time is 4 hours.

1.1 An infamous function defined on the interval [0.1] is defined by specifying its value at each rational number as 1 and its value at each irrational number as 0. What specific and serious problem with the Riemann integral does the attempt to integrate this function bring to daylight? Note: a good answer to this question does not need to be long. It just needs to be to the point.

1.2 Give the definition for an outer measure. Give the definition for Lebesgue outer measure on $\mathbf{R}$ in particular. Show that the Lebesgue outer measure of an interval (closed, open, or half-open – take your pick) is equal to its length.

1.3 Give the definition that a subset of the real numbers is (Lebesgue) measurable. Show that the union of two given Lebesgue measurable sets is Lebesgue measurable.

1.4 Define what is a measure space, and

- Give the several equivalent definitions which an extended real-valued function defined upon that measure space must satisfy in order to be measurable.
- Define the integral of a non-negative measurable function f on an arbitrary measurable subset E of a measure space X.
- State and prove Fatou's Lemma, in this context.

1.5 True or false? Prove, disprove, or provide a counterexample, as appropriate:

- Let f be a nondecreasing function defined upon a closed interval. Then the set of points at which f fails to be continuous is countable.
- One of the problems with the Riemann integral is that the integral of a monotone function may fail to exist.
- Let F be continuous on $[a, b]$ and differentiable almost everywhere. Then $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$.
- If $\{f_n\}$ is a sequence of continuous functions converging to 0 at each $x \in [0, 1]$, then $\int_0^1 f_n(x)\,dx$ can not remain always greater than 1.

2.1 Define what is a normed linear space, and what is Banach space. Show that if the norm to be used on $C[0,1]$ is defined by $\|f\| = \int_0^1 |f(x)|\,dx$ then with this norm $C[0,1]$ is a normed linear space. Further, either show that it is a Banach space with this norm, or give an example which shows it is not.

2.2 For $1 \leq p \leq \infty$, show that $C[0,1]$ is a subspace of $L^p[0,1]$. For which values of p, if any, is $C[0,1]$ a <u>dense</u> subspace? (counterexample required if it is not; proof required if it is)

2.3 Is there any inclusion or nesting relationship between $L^p[0,1]$ and $L^q[0,1]$? If so, then under what conditions upon p and q? Can anything be said along these lines if the underlying interval is not $[0,1]$ but is $[0,\infty)$ instead? Explain.

2.4 Let X and Y be normed linear spaces. Define what is meant by saying that f is a linear operator from X to Y. Also define what is meant by saying that f is a linear functional. Prove the following result.

Show that the following are equivalent for a linear operator $f : X \to Y$

- f is a continuous function (meaning, continuous at every $x \in X$).
- f is continuous at 0_X.
- f is continuous at any given, particular $x \in X$.
- There is $M > 0$ such that $\sup_{x \neq 0} \frac{\|f(x)\|}{\|x\|} \leq M$ Note: In order to avoid tiny subscripts which can cause eyestrain, $\|x\|$ obviously means here $\|x\|_X$, and $\|f(x)\|$ means $\|f(x)\|_Y$.

2.5 Show that if X is a normed linear space of finite dimension n, then every linear functional defined upon X is continuous. Also give an example to show that if X is infinite dimensional there can exist a discontinuous linear functional.

2.6 Let φ be a bounded linear functional defined on $L^1[0,1]$. Show that the function g given by $g(x) = \varphi(\chi_{[0,x]})$ is absolutely continuous on $[0,1]$.

3.1 Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$.

- Define, in terms of the two metrics, what it means for f to be continuous, also what is meant if f is uniformly continuous. Show that f is continuous if and only if the inverse image of every open set in Y is an open set in X.
- Define what it means for f to be a homeomorphism between the two spaces. Prove, or give a counterexample to the proposition that f is a homeomorphism if it is one-to-one, onto, and continuous.
- What is meant by saying that a metric space is complete? Prove, or give a counterexample to the proposition that if X is complete and f is a homeomorphism, then Y is complete.

3.2 Define what it means to say that a topological set X is connected, and show that an interval $[a, b]$ is a connected subset of the real numbers, under the topology induced upon it by the usual topology on the real numbers.

3.3 We did not study "space-filling" curves, but for the purpose of this question let us assume that we could encounter a continuous one-to-one function, say from $[-1, 1]$ onto the rectangle $[-1, 1] \times [-1, 1]$. Is it possible that any such "space-filling" function could be a homeomorphism between the interval and the rectangle, with the usual topology being in use on each? Why or why not?

4.1 Part (a)

Give the definition for the Fourier series of a 2π-periodic function, and show that (each term in the) Fourier series for a function f can be constructed if $f \in L^p(2\pi)$, whenever $p \geq 1$.

Part (b)

Show that the finite Fourier series $S_n f$ (which stops at some index n) can be written in closed form, using an integral kernel representation, as

$$S_n f(x) = \int_{-\pi}^{\pi} D_n(x,t) f(t)\, dt$$

where $D_n(x,t)$ is the Dirichlet kernel (this means, pretty much, that you should derive the closed-form expression for the Dirichlet kernel).

4.2 Given $\alpha \in (0,\infty)$, show that the function $(x,y) \mapsto e^{-\alpha xy} \sin(x)$ is Lebesgue-integrable on $(0,\infty) \times (1,\infty)$. Compute the two iterated integrals and use the result to compute

$$\int_0^\infty e^{-\alpha x} \frac{\sin(x)}{x}\, dx.$$

Hint. You may use without proof that, given $\beta \in \mathbf{R}$, an antiderivative for $x \mapsto e^{-\beta x} \sin(x)$ is given by $x \mapsto -\frac{e^{-\beta x}}{1+\beta^2}(\beta \sin(x) + \cos(x))$.