Ordinary Differential Equations (Math 7280/90) **Time:** 15:00-18:00, June 6th

Prelim Exam 2025 **Classroom:** Parker Hall 250

Auburn University
Auburn, AL

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Instructions:

1. Two textbooks, [CL] and [HSD] (see below), are permitted during the examination. Additionally, lecture notes and homework materials are allowed. However, the use of any electronic devices is strictly prohibited.
2. Please work out the problems in the space provided and show your answers clearly and legibly. You will be provided draft papers, which won't be graded.
3. Coverage: The following chapters will be tested in this exam:

CL: Chapt. 1	Existence and uniqueness of solutions
CL: Chapt. 2	Existence and uniqueness of solutions (continued)
CL: Chapt. 3	Linear differential equations (LDEs)
CL: Chapt. 4	Linear D.E.'s with isolated singularities: singularities of first kind
CL: Chapt. 5	Linar D.E.'s with isolated singularities: singularities of second kind
CL: Chapt. 13	Asymptotic behavior of nonlinear system: stability
CL: Chapt. 14	Perturbation of systems having a periodic sol. (first two sections)
CL: Chapt. 15	Perturbation theory of 2-d real autonomous systems
HSD: Chapt. 8	Equilibria in nonlinear systems
HSD: Chapt. 9	Global nonlinear techniques
HSD: Chapt. 10	Closed orbits and limit sets

Textbooks:

CL "*Theory of ordinary differential equations*", by Coddington, Earl A. and Levinson, Norman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

HSD "*Differential equations, dynamical systems, and an introduction to chaos*", by Hirsch, Moris W., Smale, Stephen, and Devaney, Robert L., Elsevier/Academic Press, Amsterdam, (Second Edition) 2004.

Mark: (out of 100) ______________

Question 1 (40 points) Consider the following second-order equation:

$$x'' + \left(1 - \frac{a}{z^2}\right) x = 0, \quad z \in \mathbb{C}.$$

1. Write the equation in the form of a first-order system. Find out all singularities on the extended complex plain and classify them as either a regular singular point or an irregular singular point.
2. Find the formal solution of the system in the sense of Theorem 2.1 on p. 142 of Coddington's book. Denote this solution by $\widehat{\Phi}(z) = \left(\widehat{\phi}_1(z), \widehat{\phi}_2(z)\right)$.
3. For each $\widehat{\phi}_i(z)$, $i = 1, 2$, find the right sector S_i and solve the equation in S_i by finding actual solution $\phi_i(z)$ such that

$$\phi_i(z) \sim \widehat{\phi}_i(z) \quad \text{as } z \to \infty.$$

This problem is presented as the example problem in §1 of Chapter 5 in Coddington's book. To solve this problem, it is necessary to have a thorough understanding of the entire chapter, as well as the preceding chapters.

Solution. **Part 1:** The given equation is

$$x'' + \left(1 - \frac{a}{z^2}\right) x = 0, \quad z \in \mathbb{C}. \tag{1}$$

Let $x_1 = x$, $x_2 = x'$. Then $x_1' = x_2$ and $x_2' = -\left(1 - \frac{a}{z^2}\right) x_1$. Thus, the system is

$$\frac{d}{dz}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\left(1 - \frac{a}{z^2}\right) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The singularities for this system include $z = 0$ and ∞.

- At $z = 0$: The coefficient $-\frac{a}{z^2}$ has a pole of order 2, so $z = 0$ is a irregular singular point.
- At $z = \infty$: As $z \to \infty$, the coefficient approaches -1, which is not a zero. Hence, by Theorem 6.1 of Chapter 4 on p. 128, $z = \infty$ is an irregular singular point.

Part 2: Find the formal solution at infinity. The system (1) can be rewritten as

$$x' = \left(A_0 + z^{-2} A_2\right) x,$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}.$$

It is clear that A_0 has two distinct eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. Then according to Theorem 2.1 of Chapter 5 on p. 142 with $r = 0$, we see that the formal solution matrix takes the form

$$\widehat{\Phi}(z) = P z^R e^Q,$$

where P is a formal power series in z^{-1},

$$P = \sum_{k=0}^{\infty} z^{-k} P_k, \quad \det P_0 \neq 0,$$

R is a diagonal matrix with complex constants, and Q is given by

$$Q(z) = zQ_0 \quad \text{with} \quad Q_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \text{i.e,} \quad Q(z) = \begin{pmatrix} iz & 0 \\ 0 & -iz \end{pmatrix}.$$

Now we will determine the matrix P and the diagonal matrix R.

Step a: Diagonalizing A_0 and constructing P_0. Let S be the matrix of eigenvectors:

$$S = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad S^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Then

$$S^{-1} A_0 S = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Denote $\widehat{A}_0 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Then, (1) becomes

$$x' = \left(S\widehat{A}_0 S^{-1} + z^{-2} A_2 \right) x.$$

By set $y = S^{-1}x$, in the y-coordinates, the system becomes

$$y' = \left(\widehat{A}_0 + z^{-2} B \right) y, \tag{2}$$

where

$$\widehat{A}_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = S^{-1} A_2 S = \frac{a}{2} \begin{pmatrix} -i & -i \\ i & i \end{pmatrix}.$$

Step b: Computing R. From Eq. (2.17) on p. 146, we see that $R = \text{diag}(A_{r+1})$. Since $r = 0$, and $A_1 \equiv 0$. We see that $R = 0$. Hence, the factor z^R is an identity matrix.

Step c: Formal solution in the diagonalized system. By previous two steps, it reduces to find a formal solution of the form for (2)

$$y(z) = \left(I + \sum_{k=1}^{\infty} z^{-k} Y_k \right) e^{z\widehat{A}_0}.$$

Let $Y_0 = I$. The recursion for Y_k is determined by substituting into the equation:

$$\begin{aligned}
y' &= \left(I + \sum_{k=1}^{\infty} z^{-k} Y_k \right)' e^{z\widehat{A}_0} + \left(I + \sum_{k=1}^{\infty} z^{-k} Y_k \right) \widehat{A}_0 e^{z\widehat{A}_0} \\
&= \left(-\sum_{k=1}^{\infty} k z^{-k-1} Y_k + \left(I + \sum_{k=1}^{\infty} z^{-k} Y_k \right) \widehat{A}_0 \right) e^{z\widehat{A}_0} \\
&= \left(\widehat{A}_0 + \sum_{k=1}^{\infty} z^{-k} Y_k \widehat{A}_0 - \sum_{k=2}^{\infty} (k-1) z^{-k} Y_{k-1} \right) e^{z\widehat{A}_0} \\
&= \left(\widehat{A}_0 + z^{-1} Y_1 \widehat{A}_0 + \sum_{k=2}^{\infty} z^{-k} \left(Y_k \widehat{A}_0 - (k-1) Y_{k-1} \right) \right) e^{z\widehat{A}_0}.
\end{aligned}$$

On the other hand, from (2), we have

$$\begin{aligned}
\left(\widehat{A}_0 + z^{-2}B\right) y(z) &= \left(\widehat{A}_0 + z^{-2}B\right)\left(I + \sum_{k=1}^{\infty} z^{-k}Y_k\right) e^{z\widehat{A}_0} \\
&= \left(\widehat{A}_0\left(I + \sum_{k=1}^{\infty} z^{-k}Y_k\right) + z^{-2}B\left(I + \sum_{k=1}^{\infty} z^{-k}Y_k\right)\right) e^{z\widehat{A}_0} \\
&= \left(\widehat{A}_0 + \sum_{k=1}^{\infty} z^{-k}\widehat{A}_0 Y_k + z^{-2}B + \sum_{k=3}^{\infty} z^{-k}BY_{k-2}\right) e^{z\widehat{A}_0} \\
&= \left(\widehat{A}_0 + z^{-1}\widehat{A}_0 Y_1 + z^{-2}\left(\widehat{A}_0 Y_2 + B\right) + \sum_{k=3}^{\infty} z^{-k}(\widehat{A}_0 Y_k + BY_{k-2})\right) e^{z\widehat{A}_0}.
\end{aligned}$$

Now equate both sides and collect powers of z^{-k} to see that

1. When $k = 0$,
$$\widehat{A}_0 = \widehat{A}_0.$$
2. When $k = 1$,
$$Y_1\widehat{A}_0 = \widehat{A}_0 Y_1, \quad \text{i.e.,} \quad Y_1 = \widehat{A}_0 Y_1 \widehat{A}_0^{-1}.$$
Since $\widehat{A}_0$ is diagonalizable, Y_1 has to be a diagonal matrix as well.
3. When $k = 2$,
$$\widehat{A}_0 Y_2 + B = Y_2\widehat{A}_0 - Y_1.$$
4. When $k \geqslant 3$,
$$\widehat{A}_0 Y_k + BY_{k-2} = Y_k\widehat{A}_0 - (k-1)Y_{k-1}.$$

Using the Lie bracket notation $[A, B] := AB - BA$, we can summarize the recursion as follows:

$$\begin{aligned}
[Y_1, \widehat{A}_0] &= 0, \\
[Y_2, \widehat{A}_0] - Y_1 &= B, \\
[Y_k, \widehat{A}_0] - (k-1)Y_{k-1} &= BY_{k-2}, \quad k \geqslant 3,
\end{aligned}$$

with

$$\widehat{A}_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad B = \frac{a}{2}\begin{pmatrix} -i & -i \\ i & i \end{pmatrix}.$$

Now solve the recursion. Make the explicit compute computation for Y_k, with $k = 1, 2$, which will require computation of $k = 3$. Let $Y_k = \begin{pmatrix} p_k & q_k \\ r_k & s_k \end{pmatrix}$ for each k.

Step c-1: Compute Y_1. We have $[Y_1, \widehat{A}_0] = 0$. Compute:

$$[Y_1, \widehat{A}_0] = Y_1\widehat{A}_0 - \widehat{A}_0 Y_1 = \begin{pmatrix} 0 & 2iq_1 \\ -2ir_1 & 0 \end{pmatrix}.$$

So $q_1 = r_1 = 0$, and Y_1 is diagonal. Let $Y_1 = \begin{pmatrix} p_1 & 0 \\ 0 & s_1 \end{pmatrix}$.

Step c-2: Compute Y_2. The recursion is $[Y_2, \widehat{A}_0] - Y_1 = B$. For $Y_2 = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}$, compute $[Y_2, \widehat{A}_0]$:

$$[Y_2, \widehat{A}_0] = Y_2\widehat{A}_0 - \widehat{A}_0 Y_2 = \begin{pmatrix} 0 & 2iq_2 \\ -2ir_2 & 0 \end{pmatrix}.$$

So

$$\begin{pmatrix} 0 & 2iq_2 \\ -2ir_2 & 0 \end{pmatrix} - \begin{pmatrix} p_1 & 0 \\ 0 & s_1 \end{pmatrix} = \frac{a}{2}\begin{pmatrix} -i & -i \\ i & i \end{pmatrix}.$$

Equating entries gives:

$$\begin{aligned}
(1,1): &\quad 0 - p_1 = -\frac{a}{2}i \implies p_1 = \frac{a}{2}i, \\
(1,2): &\quad 2iq_2 - 0 = -\frac{a}{2}i \implies q_2 = -\frac{a}{4}, \\
(2,1): &\quad -1ir_2 - 0 = \frac{a}{2}i \implies r_2 = -\frac{a}{4}, \\
(2,2): &\quad 0 - s_1 = \frac{a}{2}i \implies s_1 = -\frac{a}{2}i.
\end{aligned}$$

Thus,

$$Y_1 = \frac{a}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} p_2 & -\frac{a}{4} \\ -\frac{a}{4} & s_2 \end{pmatrix},$$

with p_2 and s_2 to be determined in the next step.

Step c-3: Compute Y_3. The recursion is $[Y_3, \widehat{A}_0] - 2Y_2 = BY_1$. First, compute BY_1:

$$BY_1 = \frac{a}{2}\begin{pmatrix} -i & -i \\ i & i \end{pmatrix}\begin{pmatrix} \frac{a}{2}i & 0 \\ 0 & -\frac{a}{2}i \end{pmatrix} = \frac{a^2}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Recall that $Y_3 = \begin{pmatrix} p_3 & q_3 \\ r_3 & s_3 \end{pmatrix}$ and $Y_2 = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}$. The commutator is

$$[Y_3, \widehat{A}_0] = \begin{pmatrix} 0 & 2iq_3 \\ -2ir_3 & 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} 0 & 2iq_3 \\ -2ir_3 & 0 \end{pmatrix} - 2\begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix} = \frac{a^2}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Equating entries gives:

$$\begin{aligned}
(1,1): &\quad 0 - 2p_2 = \frac{a^2}{4} \implies p_2 = -\frac{a^2}{8}, \\
(1,2): &\quad 2iq_3 - 2q_2 = -\frac{a^2}{4} \implies q_3 = \frac{1}{2i}\left(-\frac{a^2}{4} + 2q_2\right), \\
(2,1): &\quad -2ir_3 - 2r_2 = -\frac{a^2}{4} \implies r_3 = -\frac{1}{2i}\left(-\frac{a^2}{4} + 2r_2\right), \\
(2,2): &\quad 0 - 2s_2 = \frac{a^2}{4} \implies s_2 = -\frac{a^2}{8}.
\end{aligned}$$

Recall from previous step $q_2 = -\frac{a}{4}$, $r_2 = -\frac{a}{4}$, substituting into the expressions for q_3 and r_3 gives:

$$q_3 = \frac{1}{2i}\left(-\frac{a^2}{4} + 2\cdot\left(-\frac{a}{4}\right)\right) = \frac{1}{2i}\left(-\frac{a^2}{4} - \frac{a}{2}\right) = -\frac{1}{2i}\left(\frac{a^2}{4} + \frac{a}{2}\right) = \frac{i}{2}\left(\frac{a^2}{4} + \frac{a}{2}\right),$$
$$r_3 = -\frac{1}{2i}\left(-\frac{a^2}{4} + 2\cdot\left(-\frac{a}{4}\right)\right) = -\frac{1}{2i}\left(-\frac{a^2}{4} - \frac{a}{2}\right) = \frac{1}{2i}\left(\frac{a^2}{4} + \frac{a}{2}\right) = -\frac{i}{2}\left(\frac{a^2}{4} + \frac{a}{2}\right).$$

Thus,

$$Y_2 = \begin{pmatrix} -\frac{a^2}{8} & -\frac{a}{4} \\ -\frac{a}{4} & -\frac{a^2}{8} \end{pmatrix} \quad \text{and} \quad Y_3 = \begin{pmatrix} p_3 & \frac{i}{2}\left(\frac{a^2}{4} + \frac{a}{2}\right) \\ -\frac{i}{2}\left(\frac{a^2}{4} + \frac{a}{2}\right) & s_3 \end{pmatrix}.$$

Step c-4: Compute Y_4. The recursion is $[Y_4, \widehat{A}_0] - 3Y_3 = BY_2$. First, compute BY_2:

$$\begin{aligned} BY_2 &= \frac{a}{2}\begin{pmatrix} -i & -i \\ i & i \end{pmatrix}\begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix} \\ &= \frac{a}{2}\begin{pmatrix} -ip_2 - ir_2 & -iq_2 - is_2 \\ ip_2 + ir_2 & iq_2 + is_2 \end{pmatrix}. \end{aligned}$$

Recall $p_2 = s_2 = -\frac{a^2}{8}$, $q_2 = r_2 = -\frac{a}{4}$. So,

$$\begin{aligned} -ip_2 - ir_2 &= -i\left(-\frac{a^2}{8}\right) - i\left(-\frac{a}{4}\right) = \frac{a^2}{8}i + \frac{a}{4}i, \\ -iq_2 - is_2 &= -i\left(-\frac{a}{4}\right) - i\left(-\frac{a^2}{8}\right) = \frac{a}{4}i + \frac{a^2}{8}i, \\ ip_2 + ir_2 &= i\left(-\frac{a^2}{8}\right) + i\left(-\frac{a}{4}\right) = -\frac{a^2}{8}i - \frac{a}{4}i, \\ iq_2 + is_2 &= i\left(-\frac{a}{4}\right) + i\left(-\frac{a^2}{8}\right) = -\frac{a}{4}i - \frac{a^2}{8}i. \end{aligned}$$

Therefore,

$$BY_2 = \frac{a}{2}\begin{pmatrix} \frac{a^2}{8}i + \frac{a}{4}i & \frac{a}{4}i + \frac{a^2}{8}i \\ -\frac{a^2}{8}i - \frac{a}{4}i & -\frac{a}{4}i - \frac{a^2}{8}i \end{pmatrix} = \frac{a}{2}i\begin{pmatrix} \frac{a^2}{8} + \frac{a}{4} & \frac{a}{4} + \frac{a^2}{8} \\ -\frac{a^2}{8} - \frac{a}{4} & -\frac{a}{4} - \frac{a^2}{8} \end{pmatrix}.$$

Now, as before, the commutator is $[Y_4, \widehat{A}_0] = \begin{pmatrix} 0 & 2iq_4 \\ -2ir_4 & 0 \end{pmatrix}$. So,

$$\begin{pmatrix} 0 & 2iq_4 \\ -2ir_4 & 0 \end{pmatrix} - 3\begin{pmatrix} p_3 & q_3 \\ r_3 & s_3 \end{pmatrix} = BY_2,$$

or equivalently,

$$\begin{aligned} &(1,1): \quad 0 - 3p_3 = \frac{a}{2}i\left(\frac{a^2}{8} + \frac{a}{4}\right) \implies p_3 = -\frac{a}{6}i\left(\frac{a^2}{8} + \frac{a}{4}\right), \\ &(1,2): \quad 2iq_4 - 3q_3 = \frac{a}{2}i\left(\frac{a}{4} + \frac{a^2}{8}\right) \implies q_4 = \frac{1}{2i}\left(\frac{a}{2}i\left(\frac{a}{4} + \frac{a^2}{8}\right) + 3q_3\right), \\ &(2,1): \quad -2ir_4 - 3r_3 = -\frac{a}{2}i\left(\frac{a^2}{8} + \frac{a}{4}\right) \implies r_4 = -\frac{1}{2i}\left(-\frac{a}{2}i\left(\frac{a^2}{8} + \frac{a}{4}\right) + 3r_3\right), \\ &(2,2): \quad 0 - 3s_3 = -\frac{a}{2}i\left(\frac{a}{4} + \frac{a^2}{8}\right) \implies s_3 = \frac{a}{6}i\left(\frac{a}{4} + \frac{a^2}{8}\right). \end{aligned}$$

Hence, we obtain the complete expression for Y_3:

$$Y_3 = \begin{pmatrix} -\frac{a^2}{8} & \frac{i}{2}\left(\frac{a^2}{4}+\frac{a}{2}\right) \\ -\frac{i}{2}\left(\frac{a^2}{4}+\frac{a}{2}\right) & -\frac{a^2}{8} \end{pmatrix}.$$

In summary, we have obtained the following explicit forms for Y_1, Y_2, and Y_3:

$$Y_1 = \frac{a}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$Y_2 = -\frac{a}{4}\begin{pmatrix} \frac{a}{2} & 1 \\ 1 & \frac{a}{2} \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} -\frac{a^2}{8} & \frac{i}{2}\left(\frac{a^2}{4}+\frac{a}{2}\right) \\ -\frac{i}{2}\left(\frac{a^2}{4}+\frac{a}{2}\right) & -\frac{a^2}{8} \end{pmatrix}.$$

Step c-5: Find the formal solution. From the above Y_1, Y_2, and Y_3, we can write the formal solution as

$$\begin{aligned} \widehat{\Phi}(z) &= S\left(I + z^{-1}Y_1 + z^{-2}Y_2 + z^{-3}Y_3\right) e^{\hat{A}_0 z} + O(z^{-4}) \\ &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\left(I + z^{-1}Y_1 + z^{-2}Y_2 + z^{-3}Y_3\right) e^{\hat{A}_0 z} + O(z^{-4}). \end{aligned}$$

After tedious computations, we see that (set $a = -1/4$ and then compare the entries $\widetilde{\Phi}_{11}(z)$ and $\widetilde{\Phi}_{12}(z)$ with the last display on p. 146)

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$$\widehat{\Phi}(z) = \begin{pmatrix} e^{iz}\left[1 + \frac{ai}{2z} + \left(-\frac{a^2}{8} - \frac{a}{4}\right)\frac{1}{z^2} + \left(-\frac{a^2}{8} - \frac{i}{2}\left(\frac{a^2}{4} + \frac{a}{2}\right)\right)\frac{1}{z^3}\right] & e^{-iz}\left[1 - \frac{ai}{2z} + \left(-\frac{a^2}{8} - \frac{a}{4}\right)\frac{1}{z^2} + \left(-\frac{a^2}{8} + \frac{i}{2}\left(\frac{a^2}{4} + \frac{a}{2}\right)\right)\frac{1}{z^3}\right] \\ e^{iz}\left[i + \frac{a}{2z} + \left(\frac{ia}{4} - \frac{ia^2}{8}\right)\frac{1}{z^2} + \left(-\frac{ia^2}{8} + \frac{1}{2}\left(\frac{a^2}{4} + \frac{a}{2}\right)\right)\frac{1}{z^3}\right] & e^{-iz}\left[-i + \frac{a}{2z} + \left(\frac{ia}{4} - \frac{ia^2}{8}\right)\frac{1}{z^2} + \left(-\frac{ia^2}{8} - \frac{1}{2}\left(\frac{a^2}{4} + \frac{a}{2}\right)\right)\frac{1}{z^3}\right] \end{pmatrix} + O\left(z^{-4}\right).$$

Part 3: The right sector S_k, $k = 1, 2$. According to (5.4) on p. 162, we need to solve

$$\Re\left(i - (-i)\right)z) = 0.$$

Hence, the sections S_k, $k = 1, 2$, can be

$$\text{either} \quad \{z \in \mathbb{C} : \arg(z) \in (0, \pi)\} \quad \text{or} \quad \{z \in \mathbb{C} : \arg(z) \in (-\pi, 0)\}.$$

According to Theorem 5.1 on p. 163, we see that the asymptotic expansions in Part 2 hold in the above two cones. Then, one can carry out the analytic continuity arguments as in p. 166 (left for you to complete) to conclude that

$$x_1(z) \sim \widehat{\Phi}_{11}(z) \quad \text{in } (-\pi + \delta \leqslant \arg z \leqslant 2\pi - \delta)$$

and

$$x_2(z) \sim \widehat{\Phi}_{12}(z) \quad \text{in } (-2\pi + \delta \leqslant \arg z \leqslant \pi - \delta).$$

This completes the solution of the problem. □

Question 2 (30 points) Consider the damped pendulum system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -b\sin(x_1) - ax_2, \end{aligned}$$

where a and b are positive constants. Show that for any solution $\phi(t) = (\phi_1(t), \phi_2(t))$, there is an integer k such that

$$\phi_1(t) \to k\pi \quad \text{and} \quad \phi_2(t) \to 0 \quad \text{as } t \to \infty.$$

Distinguish between the nature of the orbits in the vicinity of $(k\pi, 0)$ for the cases k even and k odd. Sketch the orbits in the (x_1, x_2)–plane.

This problem is from the Coddington's book, p. 402, Problem 3.

Solution. Consider the Lyapunov (energy) function

$$L(x_1, x_2) = b\,(1 - \cos(x_1)) + \frac{1}{2}\,x_2^2 \;\geqslant\; 0.$$

Along solutions of the system

$$x_1' = x_2, \qquad x_2' = -b\sin(x_1) - ax_2,$$

we have

$$\begin{aligned} \dot{L} &= b\sin(x_1)\cdot x_1' + x_2 \cdot x_2' \\ &= b\sin(x_1)\cdot x_2 + x_2\,(-b\sin(x_1) - ax_2) \\ &= -a\,x_2^2 \;\leqslant\; 0, \end{aligned}$$

with equality if and only if $x_2 = 0$.

Case I: If $x_2(t) \equiv 0$, then $\dot{L} \equiv 0$. Hence, $L(x_1, 0) = b\,(1 - \cos(x_1)) = C$. Moreover, the ODE reduces to

$$x_1' = 0, \qquad 0 = x_2' = -b\sin x_1.$$

Hence, $x_1(t) \equiv k\pi$ for some $k \in \mathbb{N}$.

Case II: If $x_2(t) \not\equiv 0$, then $L(x_1, x_2)$ is nonincreasing and bounded below, so $L(x_1, x_2)$ converges as $t \to \infty$. Let $\Lambda \geqslant 0$ denote its limit, namely, $\Lambda = \lim_{t\to\infty} L(t)$. As a consequence, $\dot{L}(t) \to 0$ as $t \to \infty$, which implies that $x_2(t) \to 0$ as $t \to \infty$. Therefore, as in case I, in the limit $x_1(t) \to k\pi$, as $t \to \infty$, for some $k \in \mathbb{N}$.

Nature of the equilibria (linearization). The Jacobian at $(k\pi, 0)$ is

$$J_k = \begin{pmatrix} 0 & 1 \\ -b\cos(k\pi) & -a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b(-1)^k & -a \end{pmatrix}.$$

- <u>k even:</u> $\cos(k\pi) = 1$, characteristic equation is $\lambda^2 + a\lambda + b = 0$. The two roots are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

If $a^2 > 4b$, then we have two distinct negative roots. Hence, this equilibrium is an *improper node*. If $a^2 = 4b$, then we have two identical negative roots. Hence, this equilibrium is a *proper node*. If $a^2 < 4b$, then we have a pair of complex conjugate roots with $\Re\lambda_{1,2} = -a/2 < 0$. Hence, this gives a *spiral node*. In both cases, the equilibrium is asymptotically stable.

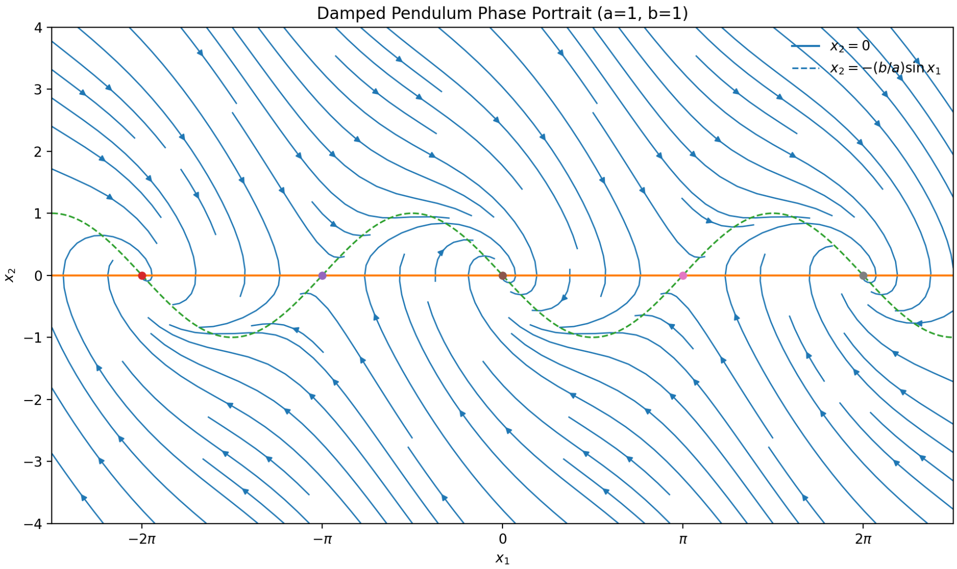
- <u>k odd:</u> $\cos(k\pi) = -1$, characteristic equation is $\lambda^2 + a\lambda - b = 0$. We have two real roots of opposite sign:

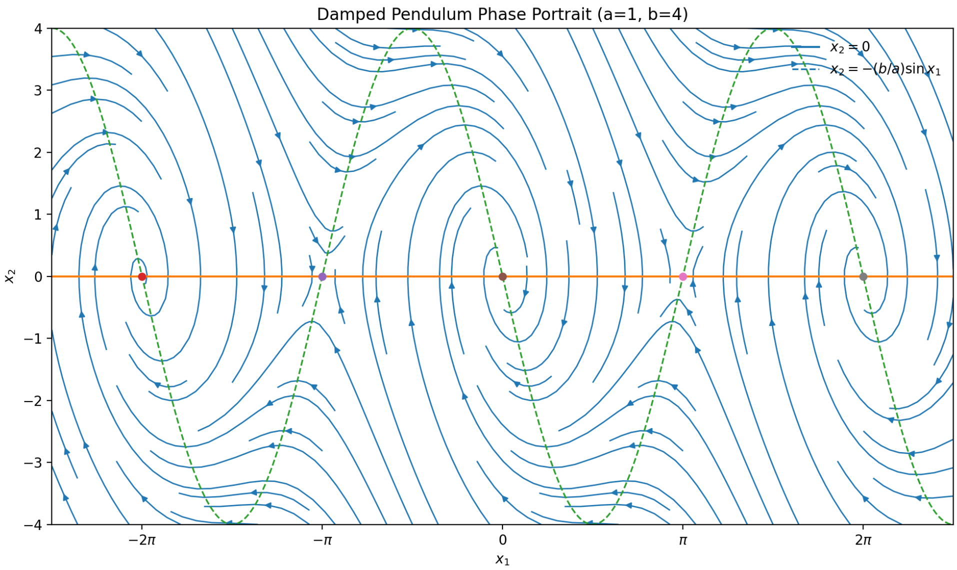
$$\lambda_1 := \frac{-a - \sqrt{a^2 + 4b}}{2} < 0 < \lambda_2 := \frac{-a + \sqrt{a^2 + 4b}}{2}.$$

  Hence, this equilibrium is a saddle point and unstable.

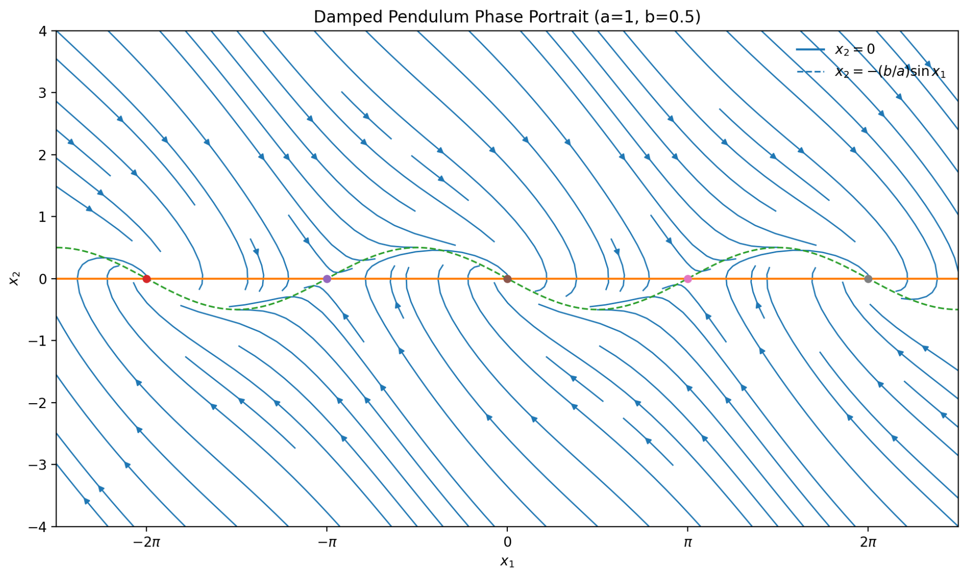
Sketch of the orbits in the (x_1, x_2)–plane:

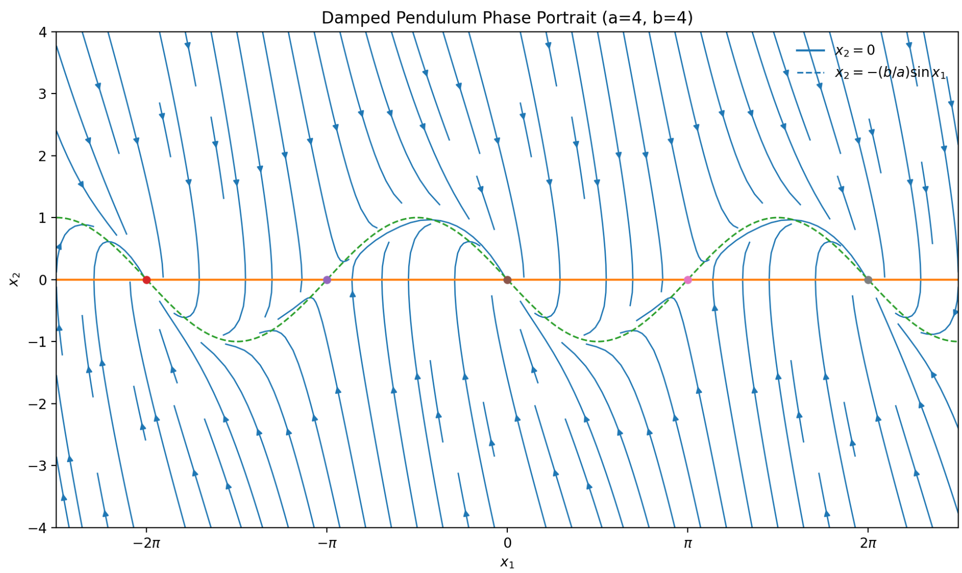
1. Case $a^2 < 4b$, *spiral nodes* at $(2k\pi, 0)$ for $k \in \mathbb{Z}$:



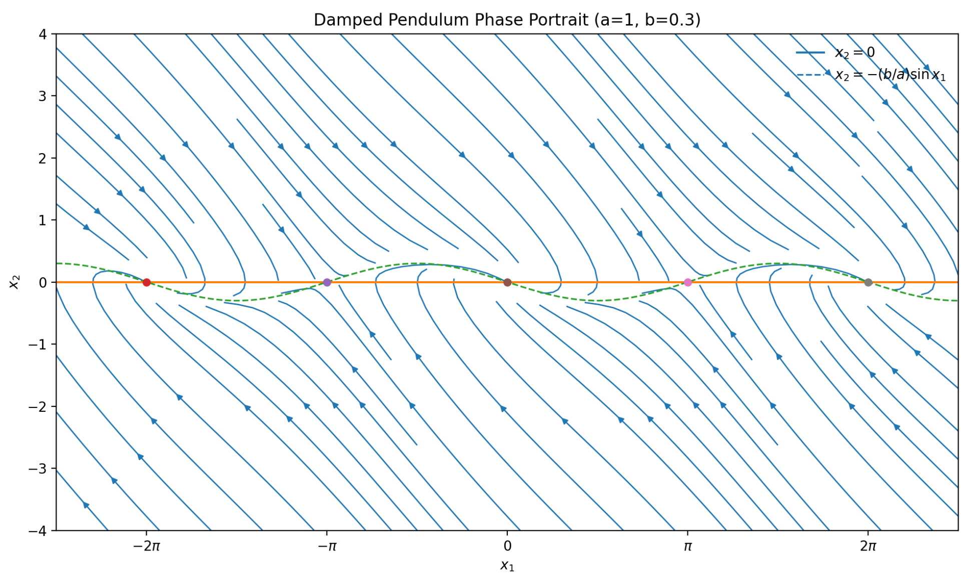


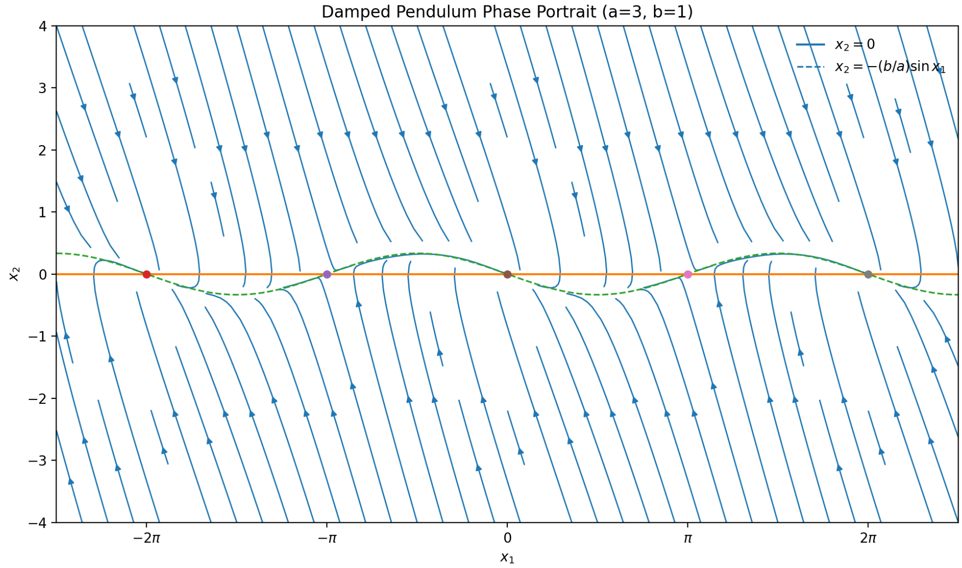
2. Case $a^2 = 4b$, *proper nodes* at $(2k\pi, 0)$ for $k \in \mathbb{Z}$:





3. Case $a^2 > 4b$, *improper nodes* at $(2k\pi, 0)$ for $k \in \mathbb{Z}$:





□

Remark *(Orbits with even k):* Fix an *even* integer k and the equilibrium $X^* = (k\pi, 0)$. Let $\xi := x_1 - k\pi$ be the perturbation of x_1 away from $k\pi$. Since k is even, $\cos(x_1) = \cos(k\pi + \xi) = \cos\xi$, and for $|\xi| \leqslant 1$, $1 - \cos\xi \geqslant \frac{1}{4}\xi^2$. Hence, in the set $|\xi| \leqslant 1$,

$$L(x_1, x_2) = b(1 - \cos\xi) + \tfrac{1}{2}x_2^2 \geqslant \tfrac{b}{4}\xi^2 + \tfrac{1}{2}x_2^2,$$

so L is *positive definite* at X^* (strict local minimum). Choose $\varepsilon > 0$ so small that the sub-level set

$$\mathcal{P} := \{(x_1, x_2) : L(x_1, x_2) \leqslant \varepsilon\}$$

lies inside $\{|\xi| \leqslant 1\}$ and contains no other equilibrium. Then $\mathcal{P}$ is closed, bounded, and positively invariant since $\dot{L} \leqslant 0$. On $\mathcal{P}$, $\dot{L} = 0$ iff $x_2 = 0$. If a solution in $\mathcal{P}$ has L constant, then $x_2 \equiv 0$ and the second equation gives $0 = x_2' = -b\sin x_1$; hence $\sin(k\pi + \xi) \equiv 0$, i.e. $\sin\xi \equiv 0$. Because $|\xi| \leqslant 1$, this forces $\xi \equiv 0$. Thus the only entire solution in $\mathcal{P}$ with L constant is the equilibrium X^*. By LaSalle's invariance principle (see [HSD]), $(k\pi, 0)$ is *asymptotically stable* and $\mathcal{P}$ lies in its basin. Note that the above arguments fail if k is odd (please figure why?).

Question 3 (30 points) In this problem, do not directly cite or rely upon the standard results of the Laplace integral method. Instead, explicitly derive the asymptotic expansions from first principles, carefully following the reasoning and key steps typical of the Laplace integral approach. Include all intermediate steps and clearly justify each part of your derivation.

Solve the following second-order equation on the complex plane:

$$zw'' + (\gamma - z)\, w' - \alpha w = 0. \tag{3}$$

Find two linearly independent solutions, denoted as $w_1(z)$ and $w_2(z)$. Determine the asymptotic behaviour of these solutions as $z \to \infty$. Additionally, identify the maximum range of z for which the asymptotic expansion remains valid.

This problem is from the Coddington's book, p. 172, Section 8. One needs to fill in significant details to complete the solution.

Solution. The ODE is in the form

$$(a_0 z + b_0)w'' + (a_1 z + b_1)w' + (a_2 z + b_2)w = 0,$$

with

$$a_0 = 1, \quad b_0 = 0, \quad a_1 = -1, \quad b_1 = \gamma, \quad a_2 = 0, \quad b_2 = -\alpha.$$

Hence,

$$\begin{aligned} P(s) &= a_0 s^2 + a_1 s + a_2 = s^2 - s, \\ Q(s) &= b_0 s^2 + b_1 s + b_2 = \gamma s - \alpha. \end{aligned}$$

Let F be an analytic function and let

$$\varphi(z) = \int_C F(s) e^{sz} \mathrm{d}s,$$

where C is a contour in the complex plane to be determined later. Suppose that φ is a solution to (3). Then, we have

$$\begin{aligned} \varphi''(z) &= \int_C F(s) s^2 e^{sz} \mathrm{d}s, \\ \varphi'(z) &= \int_C F(s) s e^{sz} \mathrm{d}s, \\ \varphi(z) &= \int_C F(s) e^{sz} \mathrm{d}s. \end{aligned}$$

Substituting these into (3) gives

$$\begin{aligned} \int_C F(s) \left[zP(s) + Q(s) \right] e^{sz} \mathrm{d}s &= \int_C F(s) \left[z(s^2 - s) + (\gamma s - \alpha) \right] e^{sz} \mathrm{d}s \\ &= z\varphi(z)'' + (\gamma - z)\, \varphi(z)' - \alpha \varphi(z) = 0. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned}\int_C F(s)zP(s)e^{sz}\mathrm{d}s &= \int_C F(s)P(s)\frac{\partial}{\partial s}e^{sz}\mathrm{d}s \\ &= F(s)P(s)e^{sz}\Big|_{\partial C} - \int_C \left(F'(s)P(s) + F(s)P'(s)\right)e^{sz}\mathrm{d}s.\end{aligned}$$

Hence,

$$\begin{aligned}\int_C F(s)\left[zP(s)+Q(s)\right]e^{sz}\mathrm{d}s =& F(s)P(s)e^{sz}\Big|_{\partial C} \\ &+ \int_C \left(F(s)Q(s) - F'(s)P(s) - F(s)P'(s)\right)e^{sz}\mathrm{d}s.\end{aligned}$$

Denote

$$V(z) := F(s)P(s)e^{sz}\Big|_{\partial C}.$$

Now, choose F so that

$$F(s)Q(s) - F'(s)P(s) - F(s)P'(s) = 0.$$

The above the first order ODE can be solved as follows:

$$\frac{F'(s)}{F(s)} = \frac{Q(s)-P'(s)}{P(s)} = \frac{\gamma s - \alpha - 2s + 1}{s^2 - s} = \frac{\gamma-2}{s-1} + \frac{1-\alpha}{s} = \frac{\alpha-1}{s} + \frac{\gamma-\alpha-1}{s-1}.$$

Hence, for some constant K,

$$F(s) = Ks^{\alpha-1}(s-1)^{\gamma-\alpha-1}.$$

Let us set $K = 1$. Therefore, φ satisfies (3) if

$$V(z) = F(s)P(s)e^{sz}\Big|_{\partial C} = s^{\alpha}(s-1)^{\gamma-\alpha}e^{sz}\Big|_{\partial C} = 0.$$

Case I: If $\Re\alpha > 0$ and $\Re(\gamma-\alpha) > 0$, then we can choose the contour C to be the real interval $[0,1]$, which ensures that $V(z) = 0$ for any $z \in \mathbb{C}$. In this case,

$$\varphi(z) = \int_0^1 s^{\alpha-1}(s-1)^{\gamma-\alpha-1}e^{sz}\mathrm{d}s, \quad \forall z \in \mathbb{C}.$$

Write

$$(s-1)^{\gamma-\alpha-1} = e^{i\pi(\gamma-\alpha-1)}(1-s)^{\gamma-\alpha-1}, \quad \text{for } s \in (0,1). \tag{4}$$

Hence,

$$\boxed{\varphi(z) = e^{i\pi(\gamma-\alpha-1)}\int_0^1 s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\,ds, \qquad \Re\alpha > 0,\ \Re(\gamma-\alpha) > 0.} \tag{5}$$

Case II: If $\Re(\gamma-\alpha) > 0$ and $\Re z < 0$, then we can choose C to be $[1,\infty)$. In this case,

$$\varphi(z) = \int_1^\infty s^{\alpha-1}(s-1)^{\gamma-\alpha-1}e^{sz}\mathrm{d}s, \quad \arg z \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

By change of variable $s = \sigma + 1$,

$$\boxed{\varphi(z) = e^z \int_0^\infty (1+\sigma)^{\alpha-1}\sigma^{\gamma-\alpha-1}e^{\sigma z}\mathrm{d}\sigma, \quad \arg z \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).} \tag{6}$$

Case III: If α, $\gamma - \alpha \neq 1, 2, 3, \dots$, then we can choose C to be the Pochhammer contour, namely, a contour that starts at $a \in (0,1)$, makes a positive makes a positive loop around 0, a positive one around 1, a negative one around 0, and a negative one around 1, and returns to a. See Figure 1. In this case,

$$\begin{aligned}\varphi(z) &= \int_a^{(0+,1+,0-,1-)} s^{\alpha-1}(s-1)^{\gamma-\alpha-1}e^{sz}\mathrm{d}s \\ &= e^{i\pi(\gamma-\alpha-1)}\int_a^{(0+,1+,0-,1-)} s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\mathrm{d}s, \quad \forall z \in \mathbb{C}.\end{aligned}$$

where we have applied (4) in the second equality.

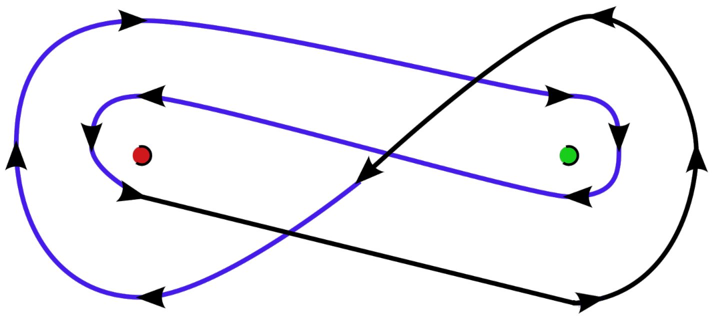


Figure 1: The Pochhammer contour C in Case III.

By letting $a = 1-$, we can reduce the Pochhammer contour integral as follows:

$$\begin{aligned}&\int_a^{(0^+,1^+,0^-,1^-)} s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\,\mathrm{d}s \\ &= e^{-2\pi i(\gamma-\alpha)}\int_1^0 s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\,\mathrm{d}s \\ &\quad + e^{-2\pi i(\gamma-\alpha)}e^{2\pi i\alpha}\int_0^1 s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\,\mathrm{d}s \\ &\quad + e^{-2\pi i(\gamma-\alpha)}e^{2\pi i\alpha}e^{+2\pi i(\gamma-\alpha)}\int_1^0 s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\,\mathrm{d}s \\ &\quad + e^{-2\pi i(\gamma-\alpha)}e^{2\pi i\alpha}e^{+2\pi i(\gamma-\alpha)}e^{-2\pi i\alpha}\int_0^1 s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\,\mathrm{d}s \\ &= (1-e^{-2\pi i(\gamma-\alpha)})(1-e^{-2\pi i\alpha})\int_0^1 s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\,\mathrm{d}s.\end{aligned}$$

Therefore,

$$\varphi(z) = \left(1-e^{-2\pi i(\gamma-\alpha)}\right)\left(1-e^{-2\pi i\alpha}\right)e^{i\pi(\gamma-\alpha-1)}\int_0^1 s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\,\mathrm{d}s, \quad \forall z \in \mathbb{C}.$$

Finally, by setting the prefactor

$$\begin{aligned}\mathcal{M}(\alpha,\gamma) &:= e^{i\pi(\gamma-\alpha-1)}\left(1-e^{-2\pi i\alpha}\right)\left(1-e^{-2\pi i(\gamma-\alpha)}\right)\\ &= 4e^{-i\pi\alpha}\sin(\pi\alpha)\sin\left(\pi(\gamma-\alpha)\right),\end{aligned}$$

we can write the solution in a more compact form:

$$\boxed{\varphi(z) = \mathcal{M}(\alpha,\gamma)\int_0^1 s^{\alpha-1}(1-s)^{\gamma-\alpha-1}e^{sz}\,ds, \quad \forall z\in\mathbb{C},} \tag{7}$$

Note that the above prefactor is nonzero under the hypothesis of Case III (no positive-integer resonance). □

Asymptotics of $\varphi(z)$ in Case I as $\Re z\to+\infty$. When $\Re z>0$, the exponential e^{sz} in (5) is largest at the endpoint $s=1$. We therefore expand the integrand near $s=1$. By change of variables,

$$t = 1-s, \quad s = 1-t, \quad \mathrm{d}s = -\,\mathrm{d}t,$$

the integral becomes

$$\begin{aligned}\varphi(z) &= e^{i\pi(\gamma-\alpha-1)}\int_0^1 (1-t)^{\alpha-1}t^{\gamma-\alpha-1}e^{(1-t)z}\,\mathrm{d}t\\ &= e^{i\pi(\gamma-\alpha-1)}e^{z}\int_0^1 (1-t)^{\alpha-1}t^{\gamma-\alpha-1}e^{-tz}\,\mathrm{d}t.\end{aligned}$$

Since e^{-tz} decays exponentially for $t\gtrsim 1/\Re z$ when $\Re z>0$, we may extend the upper limit to ∞ with an exponentially small error:

$$\varphi(z)\approx e^{i\pi(\gamma-\alpha-1)}e^{z}\int_0^\infty (1-t)^{\alpha-1}t^{\gamma-\alpha-1}e^{-tz}\,\mathrm{d}t.$$

Near $t=0$, by the Taylor expansion, we see that

$$(1-t)^{\alpha-1} = 1-(\alpha-1)t+\frac{(\alpha-1)(\alpha-2)}{2}t^2-\cdots.$$

Then we integrate term-by-term, using the definition of the Gamma function

$$\int_0^\infty t^{\beta-1}e^{-tz}\,dt = \Gamma(\beta)z^{-\beta}, \qquad \Re z>0,\ \Re\beta>0 \tag{8}$$

to see that

$$\begin{aligned}\varphi(z) &\approx e^{i\pi(\gamma-\alpha-1)}e^{z}\left[\Gamma(\gamma-\alpha)z^{\alpha-\gamma}-(\alpha-1)\Gamma(\gamma-\alpha+1)z^{\alpha-\gamma-1}+\cdots\right]\\ &\approx e^{i\pi(\gamma-\alpha-1)}e^{z}\Gamma(\gamma-\alpha)z^{\alpha-\gamma}\left[1+\frac{(1-\alpha)(\gamma-\alpha)}{z}+\cdots\right]\end{aligned}$$

Therefore, we have

$$\varphi(z)\sim e^{i\pi(\gamma-\alpha-1)}e^{z}\Gamma(\gamma-\alpha)z^{\alpha-\gamma}\left[1+\frac{(1-\alpha)(\gamma-\alpha)}{z}+\cdots\right], \quad \Re z\to+\infty.$$

□

Asymptotics of $\varphi(z)$ in Case I as $\Re z \to -\infty$. Recall, from (5), when $\Re z < 0$, the exponential e^{sz} is largest near the endpoint $s = 0_+$, hence the contribution from a neighborhood of $s = 0$ dominates. Since $\Re(-z) > 0$, $e^{sz} = e^{-s(-z)}$ decays for $s \gtrsim 1/\Re(-z)$. Therefore, from (5), we have

$$\varphi(z) = e^{i\pi(\gamma-\alpha-1)} \int_0^1 s^{\alpha-1}(1-s)^{\gamma-\alpha-1} e^{sz}\, \mathrm{d}s \approx e^{i\pi(\gamma-\alpha-1)} \int_0^\infty s^{\alpha-1}(1-s)^{\gamma-\alpha-1} e^{sz}\, \mathrm{d}s,$$

with an exponentially small error (in $|z|$) as $\Re z \to -\infty$. Near $s = 0$, by the Taylor expansion, we see that

$$(1-s)^{\gamma-\alpha-1} = 1 - (\gamma-\alpha-1)s + \frac{(\gamma-\alpha-1)(\gamma-\alpha-2)}{2} s^2 - \cdots$$

Substituting this into the integral and using the Gamma integral in (8), we obtain

$$\varphi(z) \approx e^{i\pi(\gamma-\alpha-1)}\, \Gamma(\alpha)\, (-z)^{-\alpha} \left[1 + \frac{\alpha(\alpha+1-\gamma)}{(-z)} + \cdots\right].$$

Equivalently, using $(-z)^{-\alpha} = e^{-i\pi\alpha} z^{-\alpha}$ and $(-z)^{-k} = (-1)^k z^{-k}$,

$$\varphi(z) \sim e^{i\pi(\gamma-2\alpha-1)}\, \Gamma(\alpha)\, z^{-\alpha} \left[1 - \frac{\alpha(\alpha+1-\gamma)}{z} - \cdots\right], \quad \Re z \to -\infty.$$

□

Asymptotics of $\varphi(z)$ in Case II. From (6), for all $\arg \sigma \in (-\pi, \pi)$,

$$(1+\sigma)^{\alpha-1} = 1 + (\alpha-1)\sigma + \frac{(\alpha-1)(\alpha-2)}{2}\sigma^2 + \cdots.$$

Plugging this into (6) and using the Gamma integral in (8), we have

$$\varphi(z) \sim e^z \left[\frac{\Gamma(\gamma-\alpha)}{z^{\gamma-\alpha}} + \frac{(\alpha-1)\Gamma(\gamma-\alpha+1)}{z^{\gamma-\alpha+1}} + \cdots\right] \quad \arg z \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ |z| \to \infty.$$

The integral in (6) is for $\arg \sigma = 0$. However, it converges for $\arg \sigma \in (-\pi, \pi)$. When $\arg \sigma = -\pi + \epsilon$ for some small $\epsilon > 0$, the integral in (6) is convergent for $\Re \sigma z < 0$, i.e., $\arg z \in (-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} + \epsilon)$. When $\arg \sigma = \pi - \epsilon$ for some small $\epsilon > 0$, the integral in (6) is convergent for $\Re \sigma z < 0$, i.e., $\arg z \in (\frac{3\pi}{2} - \epsilon, \frac{5\pi}{2} - \epsilon)$. Therefore, the asymptotic expansion is valid for $\arg z \in (-\frac{\pi}{2}, \frac{5\pi}{2})$. Finally,

$$\varphi(z) \sim e^z \Gamma(\gamma-\alpha) z^{\alpha-\gamma} \left[1 + \frac{(\alpha-1)(\gamma-\alpha)}{z} + \cdots\right], \quad \arg z \in \left(-\frac{\pi}{2}, \frac{5\pi}{2}\right), \ |z| \to \infty.$$

□

Asymptotics for Case III. Except the factor $\mathcal{M}(\alpha, \gamma)$, the asymptotic property of the integral is the same as Case I, where one needs to consider two cases, $\Re z > 0$ and $\Re z < 0$. □