Design Theory Prelim
January 15, 2007

1. A 4-cycle system of order *n* is a partition of the edges of K_n, each element of which induces a 4-cycle.
   a. Show that a necessary condition for the existence of a 4-cycle system of order *n* is that $n \equiv 1 \pmod 8$.
   b. Find a cyclic 4-cycle system of order 17 using difference methods.
   c. A 4-cycle system is said to be nearly-resolvable if the set of 4-cycles can be partitioned into sets, called near parallel classes, each of which contains $(n-1)/4$ vertex-disjoint cycles.
      i. How many near parallel classes would a nearly-resolvable 4-cycle system of order *n* contain?
      ii. What does this tell you about the existence of nearly resolvable 4-cycle systems?

2. Suppose that (S_1,T_1) and (S_2,T_2) are Steiner Triple Systems with S_1 a proper subset of S_2 and T_1 a subset of T_2.
   a. Show that $|S_2| \geq 2|S_1| + 1$. (Hint: Consider all the triples containing a point in S_2 / S_1.)
   b. Describe the triples in T_2 / T_1 that contain the point *p* in S_1.
   c. Find a STS(15) that contains a STS(7) (Hint: (2b) should help).

3. Let $L_1, \ldots, L_x$ be a complete set of latin squares of order *n* constructed using the finite field construction.
   a. What is the value of *x*?
   b. Which, if any, of these latin squares is unipotent (all diagonal cells contain the same symbol)? Why?
   c. How many of these latin squares have each of the *n* symbols appear in a diagonal cell? Why?
   d. Suppose $L_1, \ldots, L_x$ were constructed by defining addition and multiplication modulo *n* instead of using a finite field. If $n = 8$,
      i. Which of them would be latin squares? Why?
      ii. Would any pair be orthogonal latin squares? Why?

4. Generalize the construction for a KTS using PBDs to obtain a resolvable BIBD(v=76, *k*=4) as follows.
   a. Describe how to construct an affine plane of order 4 (that is, a resolvable BIBD(16,4)).
   b. What would change in this construction in order to make an affine plane of order 5? (In (4c), think of this as a PBD(25) with all blocks of size 5.)
   c. Describe how you can use these two ingredients to construct a resolvable BIBD(76, 4) on the vertex set $\{\infty\} \cup (\{1, \ldots, 25\} \times \{1,2,3\})$. Be sure to describe both the blocks and the parallel classes.