Design Theory prelim. – January 2012

1) We denote by (a; b, c, d) the following graph G: it has vertex set {a, b, c, d} and edge set {ab, bc, bd, cd}. (aka "triangle with a stick")

a) What are the "obvious" necessary conditions on the positive integer n for the existence of a G design on K_n?

b) Show me a Steiner triple system of order 7, and fiddle with it to produce a G design on K_8.

c) The following are base blocks for a G design on K_{24}, using difference methods (mod 23) on the vertex set $Z_{23} \cup \{\infty\}$: (∞; 0, 1, 6), (9; 0, 2, 10), (11; 0, 3, 7). For each of the following pairs {x, y}, find the unique block containing the edge xy: {2, 22}, {7, 22}, {3, 11}, {∞, 17}.

2) We define $S = (s_1, s_2, \ldots, s_{2k})$ to be a *Skolem sequence* of *order* k if the s's consist of the integers from 1 to k, each appearing twice, so that for each $1 \le i \le k$, the two occurrences of symbol i are distance i apart, i.e. separated by i-1 other symbols. For example, here is a Skolem sequence of order 5: (5, 2, 4, 2, 3, 5, 4, 3, 1, 1).

a) Find a Skolem sequence of order 4.

b) Show there is no Skolem sequence of order 6. (Hint: parity.)

If $S = (s_1, s_2, \ldots, s_{2k})$ is a Skolem sequence of order k, consider the following set B of triples: for each $1 \le i \le k$, if s_a is the first occurrence of symbol i, then the triple $\{0, i, i + k + a\}$ is a triple in B. It turns out that B is the set of base blocks for a cyclic Steiner triple system of order 6k + 1 on the set Z_{6k+1}. (You don't have to prove this, just trust me.)

c) List these base blocks for the Skolem sequence of order 5 given above. Also, for each of the following pairs {x, y}, find the unique triple containing x and y in the resulting Steiner triple system: {5, 25}, {2, 9}, {8, 29}.

OVER →

3) Let t, k, v, λ be positive integers with $t \leq k \leq v$. We say (V, B) is a *(v, k, λ) t-design* if

i) V is a set of size v,
ii) B is a collection (multi-set) of k-element subsets of V, called *blocks*, and
iii) every t-element subset of V is contained in exactly λ blocks.

a) Prove that if there is a (v, k, λ) t-design, and $t > 1$, then there is a (v-1, k-1, λ) (t-1)-design.

b) Prove that if there is a (v, k, λ) t-design, then for $0 \leq i \leq t-1$, $\binom{k-i}{t-i}$ divides $\lambda\binom{v-i}{t-i}$.

4) Two latin squares L and M of order n are said to be *orthogonal* if, for $1 \leq k, l \leq n$, there is exactly one pair (i, j) such that cell (i, j) of L contains k and cell (i, j) of M contains l.

a) Suppose R and S are latin squares of order n with the property that if two cells of R contain the same symbols then the corresponding cells in S contain different symbols. Either prove that R and S are orthogonal, or give an example to show that they may be not orthogonal.

b) If you have a complete set of orthogonal latin squares that were constructed using the finite field construction, decide how many (if any) of the latin squares have a constant main diagonal (the same symbol occurs in all diagonal cells), proving your answer.

KEY

1) c) $\{2, 22\}$: $(11; 0, 3, 7) - 1 = (10; \underline{22}, \underline{2}, 6)$
$\{7, 22\}$: $(9; 0, 2, 10) - 3 = (6; 20, \underline{22}, \underline{7})$
$\{3, 11\}$: $(9; 0, 2, 10) + 1 = (10; 1, \underline{3}, \underline{11})$
$\{\infty, 17\}$: $(\infty; 0, 1, 6) - 6 = (\underline{\infty}; \underline{17}, 18, 0)$

2) a) Any of (4, 2, 3, 2, 4, 3, 1, 1), (4, 1, 1, 3, 4, 2, 3, 2), (3, 4, 2, 3, 2, 4, 1, 1), or their reverses.

c) $\{0, 1, 5\}, \{0, 2, 9\}, \{0, 3, 13\}, \{0, 4, 12\}, \{0, 5, 11\}$

$\{5, 25\}$: $\{0, 5, 11\} - 6 = \{\underline{25}, 30, \underline{5}\}$
$\{2, 9\}$: $\{0, 2, 9\} + 0 = \{0, \underline{2}, \underline{9}\}$
$\{8, 29\}$: $\{0, 3, 13\} - 5 = \{26, \underline{29}, \underline{8}\}$