Name: ....

Convex and Discrete Geometry Prelim, May 7, 2012

Work only on 7 of the questions. Circle the one which you would like to disregard.

1. A) State the one and the two dimensional versions of Helly's theorem.

   B) A finite family of convex sets in R^2, having at least two members, is such that given any two of its sets, there is some line parallel to the x axis which meets them both. Prove the existence of a line parallel to the x axis which meets all members of the family. Suggest generalizations of this result to R^3.

2. A) State the definition of the Minkowski sum of two convex sets. Illustrate the definition by drawing the Minkowski sum of a regular hexagon and a regular triangle of the same edge length.

   B) Prove or disprove: Any regular polygon is the Minkowski sum of a finete number of line segments in the plane? Beside giving a yes or no answer try to classify the regular polygons.

3. A) State the definition of Steiner symmetrization and its basic properties.

   B) Prove that if every line through a given point cuts the area of a polygon into two equal parts, then the polygon is centrally symmetric.

4. A) State the definition of the Delone triangulation and describe how to obtain the Delone triangulation of a planar point set using a paraboloid.

   B) Prove that the smallest angle of any triangulation of a convex polygon whose vertices lie on a circle is the same for each triangulation (the solution involves some elementary geometric knowledge of the circle, chords, and angles).

5. A) State Sperner lemma.

   B) Start with a triangulation of a sphere. Label all the vertices with one of three letters A, B, C. The claim is that there is an even number of ABC triangles.

   Finish the following proof (Do not look for an argument where you say that now you use Sperner lemma and you are done. In fact by completing the proof with a common sense argument you are essentially making an alternate proof for the Sperner Lemma.).

   "We may assume there is at least one ABC triangle. Place doors on AB edges and, starting with an ABC triangle form a path through the doors. The number of triangles is finite, meaning that somewhere, i.e., in one of the triangles, the path will have to terminate...." What can you say about the last triangle? Why does it have to be different from the original triangle?

6. A) State Jensen's inequality.

   B) Prove that among all n-gons circumscribed to a circle the regular one has the smallest area.

7. A) State the definition of the Gaussian curvature $K(p)$ at a point p of a polyhedral surface. State the polyhedral Gauss-Bonnet theorem.

   B) Show that every simple connected polyhedra has at least three vertices with positive curvature. Illustrate with examples that a simple connected polyhedra can have as few as 3 vertices with positive curvature.

8. A) State the definition of the Dehn invariant and the definition of scissor congruence. Explain how can it be used to show that the regular tetrahedron cannot be scissor congruent with a cube.

   B) How can the concept of scissor congruence be used to prove the Pythagorean theorem?