Graph Theory Prelim 2017

1. In this question, assume that all graphs are simple. In each of the following, prove or disprove the assertion.
   (a) Every connected graph G has a spanning tree with the same maximum degree as G.
   (b) Every connected graph G has a spanning tree with the same minimum degree as G.
   (c) Every connected graph G has a spanning tree with the same independence number as G.
   (d) Every connected graph G has a spanning tree with the same vertex cover number (minimum number of vertices incident to every edge) as G.
   (e) Every connected graph G has a spanning tree with the same chromatic number as G.
   (f) Every connected graph G has a spanning tree with the same chromatic index (edge chromatic number) as G.
   (g) Every connected graph G has a spanning tree with the same domination number (minimum size of a set of vertices of the graph such that each vertex of the graph outside of the set is adjacent to some vertex in the set) as G.

2. Let D be a digraph.
   (a) Prove that the following are equivalent:
      (i) For every ordered pair (v, w) of vertices of D, there is a directed walk in D from v to w.
      (ii) For every ordered pair (S, T) satisfying $S \cup T = V(D)$, $S \neq \emptyset \neq T$, and $S \cap T = \emptyset$, there is it least one arc of D directed from a vertex in S to a vertex in T.

      (A digraph satisfying either (and hence both) of the above conditions is said to be *strongly connected*.)
   (b) A *strong component* of D is a maximal subdigraph of D that is strongly connected. Define the digraph D^* from D as follows. The vertices of D^* are the strong components of D. If A and B are two such strong components, then there is an arc in D^* directed from A to B if and only if $A \neq B$, and there is at least one arc in D directed from a vertex in A to a vertex in B.
      Prove that D^* is acyclic.

3. The Four Colour Theorem states that every simple planar graph is 4-colourable. Prove that each of the following statements implies the Four Colour Theorem.
   (a) Every simple connected planar graph is 4-colourable.
   (b) Every simple 2-connected planar graph is 4-colourable.
   (c) Every simple 3-connected planar graph is 4-colourable.
   (d) Every simple 3-connected triangulation is 4-colourable.
   (e) Every simple 3-edge-connected 3-regular plane graph is 4-face-colourable.

   Recall: A connected graph is k*-connected* (k*-edge-connected*) if every vertex (edge) cut has at least k vertices (edges). There is one exception to this last sentence: if $n > 1$, K_n is considered to be $(n-1)$-connected but not n-connected.

   Recall: A planar graph is a *triangulation* if every face is bounded by a triangle.

4. We know that if a graph contains K_t as a subgraph, then it's chromatic number is at least t. Consider the other direction of this statement:
$$\chi(G) \geq t \Rightarrow G \text{ contains } K_t \text{ as a subgraph.} \tag{1}$$
   (a) Provide a counterexample to show that (1) is FALSE in general.
   (b) State Brooks' Theorem and explain how it proves that a particular instance of (1) is true. *(You do not need to prove Brooks' Theorem).*
   (c) If the word "subgraph" in (1) is replaced with the word "minor", then the statement becomes *Hadwiger's Conjecture*. Explain why Hadwiger's Conjecture is true for each of $t = 1, 2, 3$.
   (d) Prove that the other direction of Hadwiger's Conjecture is FALSE for every $t > 2$. That is, prove that for every $t > 2$, there are graphs containing K_t as a minor that do not have $\chi(G) \geq t$. (In fact, it is very false, as you can describe graphs with K_t-minors that are 2-colourable).