2019 Graph Theory Prelim

1. Suppose that G is a simple graph.
   a. Show that $\Sigma_{v \text{ in } V(G)} d_G(v)$ is even.
   b. State Tutte's theorem on the existence of 1-factors in graphs.
   c. Find a 4-regular simple graph with no 1-factor.
   d. Suppose that G is 3-edge-connected, 4-regular, and has an even number of vertices. Show that G has a 1-factor (Hint: Use 1(a) and 1(b)).
2. An edge-coloring is said to be equalized if $|c_i - c_j| \leq 1$, where c_i is the number of edges colored i.
   a. Show that if G has a proper k-edge-coloring then G has an equalized proper k-edge-coloring. (Hint: If your coloring is not equalized, make it better by focusing on two suitably chosen color classes.)
   b. For **each** of the **two** following problems, model them graph theoretically. In each case, specifically:
      i. Define what the vertices and edges in your graph represent,
      ii. State which graph parameter(s) can be used to solve the problem(s), describing why that is the case,
      iii. Describe what values that parameter might take on (again, in terms of graph parameters), using theorems to substantiate your answer, and
      iv. Describe what advantage would be gained by using a coloring that is "equalized" (if you are coloring vertices, then define what that would mean).
   - Various university clubs are to meet one evening to plan their efforts to help Auburn win the Auburn-Alabama Food Fight. Each club can only meet if all of its members are present (some people are members of more than one club). Each meeting lasts 30 minutes, and meetings start at 6 pm. Clubs which meet at the same time do so in different rooms. In terms of a graph parameter, find the minimum number of time periods needed in order that the schedule allows all clubs to meet with all members present. How many rooms are needed at the busiest time?
   - Various companies each send a representative to Atlanta Airport. Each representative is to meet one-on-one with other representatives, and are alone in their room for confidentiality purposes. All companies are in the same industry, but not every company needs to hold discussions with each other company. Each meeting lasts 30 minutes, and meetings start at 10 am. In terms of a graph parameter, find the minimum number of time periods needed in order that the schedule allows for all desired meetings to take place.
3. Let S be a set of edge-disjoint 2-factors in a simple graph G, and let $E(S) = \{e \mid e \text{ is an edge in a 2-factor in } S\}$. $G - E(S)$ is said to be the complement of S in G. S is said to be a maximal if and only if $G - E(S)$ has no 2-factors. A graph G is said to be primitive if G has no d-factor for $1 \leq d < \Delta(G)$.
   a. Find a primitive cubic graph. (Hint: Tutte's theorem may be useful; the smallest one has 16 vertices).
   b. Petersen's Theorem says that there exists a 2-factorization of a graph G if and only if it is regular of even degree. Using this, show that S is a maximal set of 2-factors in a regular graph G if and only if its complement is primitive.
   c. Find a maximal set S of edge-disjoint *connected* 2-factors in K_7 (that is, the complement of S has no *connected* 2-factors; equivalently, the complement of S has no Hamilton cycles), describing why S is maximal. (You can draw the elements of S to answer this if you prefer.)
   d. Show that if S is a maximal set of edge-disjoint *connected* 2-factors in K_n then $|S| > (n-2)/4$.
4. Let G be a planar graph (which is not necessarily simple). Euler's formula for G relates the number of vertices v(G), edges e(G), faces f(G) and components c(G), as follows:

$$v(G) - e(G) + f(G) = c(G) + 1.$$

   a. Use induction to prove Euler's Formula. (Hint: Edge-contraction works well here.)
   b. The length of a face in a planar drawing of a graph is the length of the shortest closed walk bounding the face. Given any planar graph G, write an equation for e(G) in terms of the lengths of the faces in G, justifying your answer. (Hint: This is an analog of the degree-sum formula.)
   c. Use Euler's formula, and your answer in (b), to prove that K_5 is not a planar graph.