Graph Theory Prelim Summer 2023

Prelim 7/3/2023 8-11am (Time: 3 hours)

Name:

Email:

There are FOUR questions. Answer all FOUR and submit. Start a new page for each question.

1. (a) Explain why a (finite) graph G with $\delta(G) \geq 2$ must contain a cycle.

   (b) Use (a) and *induction* to prove the following result.

   Let G be a connected graph where every vertex has even degree. Then G can be decomposed into cycles.

   (c) The result proved in (b) can be expanded into a stronger theorem. State the strongest possible version of this theorem; you do not need to prove your strong theorem.

2. (a) Carefully state *Dirac's theorem*.

   (b) Let G be a graph on $n \geq 2$ vertices with minimum degree $\delta(G) \geq \frac{n-1}{2}$. Using (a), or otherwise, prove that G has a *Hamilton path* (that is, a path using all n vertices).

   (c) Must every graph G on 1001 vertices with minimum degree ≥ 500 have a Hamilton *cycle*? Prove or give a counterexample.

3. (a) State *Hall's Theorem* (in the finite case). You do not need to provide a proof.

   (b) Recall that if G is a regular bipartite graph with bipartition (A, B), then $|A| = |B|$. Use this fact and Hall's theorem to explain why regular bipartite graphs with at least one edge have perfect matchings. Is this still true for bipartite *multigraphs*, with parallel edges allowed?

   (c) Use (b) to prove the following result.

   A 3-regular graph G has an nz 3-flow iff it is bipartite.

   Definition Reminders: A *flow* on a graph G is an orientation of G and a weight function f on $E(G)$ such that for every $v \in V(G)$, the sum $f^-(v)$ of the f-values on edges pointing into v equals the sum $f^+(v)$ of the f-values on edges pointing out of v. If f is integer-valued and satisfies $|f(e)| \leq k-1$ for all $e \in E(G)$ and some integer $k \geq 2$, then the flow is called a *k-flow*. If $f(e) \neq 0$ for all $e \in E(G)$ then the flow is called *nowhere zero*, abbreviated *nz*.

   (d) Suppose $G = (V, E)$ is a bipartite graph with $V = [2n]$. Suppose $f_1, \ldots, f_k$ are edges such that , for some positive integers $t, n_1, \ldots, n_k$, we have $(t, t \ldots, t) = n_1 \mathbb{1}_{f_1} + \cdots + n_k \mathbb{1}_{f_k}$. Use (b) to prove that G has a perfect matching.

Definition Reminders: For a graph $G = ([n], E)$ and an edge $uv \in E$, its *indicator vector* $\mathbb{1}_{uv}$ is given by the vector with 1's in positions u and v and 0's everywhere else.

(e) Give an example to show that (d) is false if G is not bipartite.

4. (a) The *$s - t$ paths lemma* states the following: Let D be an $s - t$ network with k interior vertices. Then any $k + 1$ $s - t$ paths in D have a rainbow $s - t$ path. Prove the $s - t$ paths lemma.

   (b) Let G be a graph on n vertices. Show that any n cycles in G have a rainbow cycle.

   (c) Let G be a graph on 1001 vertices. Is it always true that any 1000 cycles in G have a rainbow cycle? Prove or give a counterexample.