Some Graph Theory Prelim Questions

1) If G is a simple graph with at least one edge, we define the *line graph* of G, denoted *L(G)*, as follows: the vertices of L(G) are the edges of G, and two different vertices of L(G) are adjacent if, as edges of G, they have exactly one vertex of G in common.

a) Find a formula for the number of edges of L(G) in terms of the vertex degrees of G.

b) Show that if G has a vertex of degree at least d, then L(G) contains a clique (= complete subgraph) on at least d vertices.

c) Show that if $d \geq 2$, and G has a vertex of degree at least d, then L(L(G)) has a vertex of degree at least $2d - 4$.

d) If G has a vertex of degree 4, show that L(L(L(G))) has a vertex of degree at least 6.

2) Recall that an *Euler tour* in G is a closed walk that uses each edge of G exactly once.

a) State Euler's theorem on Euler tours.

b) Form the graph G^+ by adding a new vertex to G adjacent to all the vertices of odd degree in G. Under what conditions does G^+ have an Euler tour?

c) Prove (possibly by using a) and b) above) that G has an orientation so that for every vertex v, the indegree of v and the outdegree of v differ by at most one.

3) A 2-edge-coloring of the edges of the graph G with the two colors orange and blue is defined to be *balanced* if for each vertex v, the number of orange edges incident with v and the number of blue edges incident with v differ by at most one.
A connected graph is defined to be *evil* if it has an odd number of edges, and all of its vertices have even degree.

a) Prove that an evil graph has no balanced 2-edge-coloring.

b) Prove that G has a balanced 2-edge-coloring if and only if G has no evil component. (Hint: maybe 2) a) and b) above might help.)

4) a) State Hall's theorem on matchings in bipartite graphs.

b) Let G be a bipartite graph with bipartition (A, B). Suppose that for some positive integer k, each vertex in A has degree at least k, and each vertex in B has degree at most k. Show that G has a matching saturating A.