Probability (Math 7800/10) **Time:** 15:00–18:00, Aug. 20th

Prelim Exam 2025 **Classroom:** Parker Hall 250

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Print Full (First, Last) Name: ________________________

Instructions:

1. This is a closed book exam. The use of any electronic devices is strictly prohibited.
2. Please work out the problems in the space provided and show your answers clearly and legibly. You will be provided draft papers, which won't be graded.
3. You may bring two US letter-sized sheets with notes written on both sides for reference. You may write anything you wish on these sheets; however, the use of any other materials, including books, additional notes, or electronic devices, is not permitted.
4. Coverage: The following chapters will be tested in this exam:

Billingsley's *Probability and Measure*	
§1	Borel's Normal Number Theorem
§2	Probability Measures
§3	Existence and Extension
§4	Denumerable probabilities
§5	Simple random variables
§6	The law of large numbers
§7	Gambling systems
§10	General measures
§11	Outer measures
§12	Measures in Eulidean space
§13	Measurable functions and mappings
§14	Distribution functions
§15	The Integral
§16	Properties of integral
§17	The integral with respect to Lebesgue measure
§18	Product measure and Fubini theorem
§20	Random Variables and Distributions
§21	Expected Values
§22	Sums of Independent Random Variables
§25	Weak convergence
§26	Characteristic Functions
§27	The Central Limit Theorem

Mark: (out of 100) ______________

Question 1 (15 points) On the field $\mathcal{B}_0$ in $(0, 1]$ define $\mathbb{P}(A)$ to be either 1 or 0 according as there does or does not exist some positive ϵ_A (depending on A) such that A contains the interval $\left(\frac{1}{2}, \frac{1}{2} + \epsilon_A\right]$. Show that $\mathbb{P}$ is finitely but not countably additive.

Problem 2.15 on p. 34. (Homework in 2022, not 2024)

Solution. Let $\mathcal{B}_0$ be the field of finite unions of pairwise-disjoint half-open intervals $(a, b] \subset (0, 1]$. According to the problem, $\mathbb{P}(A)$ is defined as follows:

$$\mathbb{P}(A) := \begin{cases} 1, & \text{if } \exists\, \epsilon_A > 0 \text{ with } (1/2,\ 1/2 + \epsilon_A] \subset A, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Finite additivity. It suffices to establish the property for two disjoint sets and then proceed by induction. Let $A, B \in \mathcal{B}_0$ be two disjoint sets. Since both A and B are finite unions of half-open intervals, and $A \cup B$ is also a finite union of such intervals, there are only two possible cases to consider:

Case I: For some $\varepsilon > 0$, $(1/2,\ 1/2 + \varepsilon] \subset A \cup B$. Since A and B are disjoint, there is some $\varepsilon' \in (0, \varepsilon]$ such that

$$\text{either} \quad (1/2,\ 1/2 + \varepsilon'] \subset A \quad \text{or} \quad (1/2,\ 1/2 + \varepsilon'] \subset B.$$

Hence, either $\mathbb{P}(A)$ or $\mathbb{P}(B)$ is 1 and the other is 0. Therefore,

$$\mathbb{P}(A \cup B) = 1 = \mathbb{P}(A) + \mathbb{P}(B).$$

Case II: For some $\varepsilon > 0$, $(1/2,\ 1/2 + \varepsilon] \not\subset A \cup B$. In this case, $\mathbb{P}(A \cup B) = 0$ because one cannot find $\varepsilon' > 0$ so that $(1/2,\ 1/2 + \varepsilon'] \subset A \cup B$. On the other hand, neither A nor B contains an interval to the right of $1/2$. This implies that $\mathbb{P}(A) = \mathbb{P}(B) = 0$. Therefore,

$$\mathbb{P}(A \cup B) = 0 = \mathbb{P}(A) + \mathbb{P}(B).$$

Combining both cases, we conclude that $\mathbb{P}$ is finitely additive on $\mathcal{B}_0$.

(2) Not countably additive. For $n \geq 1$, define

$$A_n := \left(1/2 + 2^{-n-1},\ 1/2 + 2^{-n}\right] \in \mathcal{B}_0.$$

These A_n are pairwise disjoint and

$$\bigcup_{n=1}^{\infty} A_n = (1/2,\ 1].$$

Each A_n does not contain any interval of the form $(1/2, 1/2 + \varepsilon]$, so $\mathbb{P}(A_n) = 0$. But the union $(1/2, 1]$ does contain such an interval (e.g. $(1/2, 3/4]$), hence

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = 1.$$

Therefore,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 \neq 0 = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

This shows that $\mathbb{P}$ is finitely but not countably additive on $\mathcal{B}_0$. $\square$

Question 2 (15 points)

1. Construct an open, dense set in $(0, 1)$ with measure near 0 (as small as you want).

2. Construct a nowhere dense set $B \in (0, 1)$ with measure near 1 (as close to 1 as you want).

Hint: The set A is by definition *dense* in the set B if for each $x \in B$ and each open interval $J \ni x$, it holds that $J \cap A \neq \emptyset$. This is the same thing as requiring that $B \subset A^-$. The set E is by definition *nowhere dense* in the set B if each open interval I contains an open interval J such that $J \cap E = \emptyset$.

Examples 3.1 and 3.2 on p. 44, which were discussed in class.

Question 3 (20 points) Consider the function $d_n(\omega)$ defined on the unit interval by the dyadic expansion

$$\omega = \sum_{n=1}^{\infty} \frac{d_n(\omega)}{2^n} = .d_1(\omega)d_2(\omega)d_3(\omega)\ldots,$$

where $d_n(\omega) \in \{0, 1\}$ is the n-th digit in the dyadic expansion of $\omega \in [0, 1]$. Let ℓ_n be the length of the run of 0's starting at $d_n(\omega)$: $\ell_n(\omega) = k$ if $d_n(\omega) = \cdots = d_{n+k-1}(\omega) = 0$ and $d_{n+k}(\omega) = 1$. Show that

$$\mathbb{P}\left(\omega : \limsup_n \frac{\ell_n(\omega)}{\log_2 n} = 1\right) = 1,$$

in the following steps:

1. (5 points) For any $r \geq 0$, show that

$$\mathbb{P}\left(\omega : \ell_n(\omega) \geq r\right) = 2^{-r},$$

2. (5 points) Let $\{r_n\}$ be a sequence of positive reals such that $\sum_{k=0}^{\infty} 2^{-r_k} < \infty$. Show that

$$\mathbb{P}\left(\omega : \ell_n(\omega) \geq r_n \text{ infinitely often}\right) = 0.$$

3. (5 points) By setting $r_n = (1 + \epsilon) \log_2 n$ with $\epsilon > 0$, deduce that

$$\mathbb{P}\left(\omega : \limsup_n \frac{\ell_n(\omega)}{\log_2 n} \leq 1\right) = 1,$$

4. (5 points) If r_n is nondecreasing and $\sum_{n=1}^{\infty} 2^{-r_n} r_n^{-1}$ diverges, show that

$$\mathbb{P}\left(\omega : \ell_n(\omega) \geq r_n \text{ infinitely often}\right) = 1.$$

Then apply this to $r_n = \log_2 n$ to show that

$$\mathbb{P}\left(\omega : \ell_n(\omega) \geq \log_2 n \text{ infinitely often}\right) = 1.$$

Finally, we can make the final conclusion from Parts 3 and 4.

Hint For part 4, define $\{n_k\}$ inductively by $n_1 = 1$ and $n_{k+1} = n_k + r_{n_k}$, $k \geq 1$. Let

$$A_k := \{\omega : \ell_{n_k}(\omega) \geq r_{n_k}\} = \{\omega : d_i(\omega) = 0 \text{ for all } n_k \leq i < n_{k+1}\}.$$

Show that $\{A_k\}$ are independent and $\sum_{k=1}^{\infty} \mathbb{P}(A_k)$ diverges. Then use the second Borel-Cantelli lemma to conclude that $\mathbb{P}(\limsup A_k) = 1$. Finally, show that $\limsup A_k \subseteq \{\omega : \ell_n(\omega) \geq r_n \text{ infinitely often}\}$.

Examples 4.11 and 4.15 on pp. 59–62, which were studied in details in class.

Question 4 (10 points) Construct an example of $X_n \to X$ in probability but not almost surely. You may follow the steps below:

1. (5 points) Let

$$A_1 = \left(0, \frac{1}{2}\right], \quad A_2 = \left(\frac{1}{2}, 1\right],$$

and

$$A_3 = \left(0, \frac{1}{4}\right], \quad A_4 = \left(\frac{1}{4}, \frac{1}{2}\right], \quad A_5 = \left(\frac{1}{2}, \frac{3}{4}\right], \quad A_6 = \left(\frac{3}{4}, 1\right].$$

Define the next eight, $A_7, \cdots, A_{14}$ as the dyadic intervals of rank 3. Let $X_n := \mathbf{1}_{A_n}$. Show that $X_n \to 0$ in probability.

2. (5 points) Show that

$$\mathbb{P}\left(\omega : X_n(\omega) \to 0,\ n \to \infty\right) = 0 \neq 1.$$

Hence, we can conclude that $X_n \to 0$ in probability but not almost surely.

Example 5.4 on p. 71, which was discussed in class.

Question 5 (15 points)

1. (5 points) State the definition of the uniform integrability of a sequence $\{f_n\}$.

2. (10 points) Let $\mu(\Omega) < \infty$ and $f_n \to f$ almost everywhere. Show that if $\{f_n\}$ is uniformly integrable, then f is integrable and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Proof of Theorem 16.14 on p. 217.

Question 6 (25 points) Suppose that for each n

$$X_{n1}, \cdots, X_{nr_n}$$

are independent. Set $S_n = X_{n1} + \cdots + X_{nr_n}$. Assume that

$$\mathbb{E}[X_{nk}] = 0, \quad \sigma_{nk}^2 = \mathbb{E}\left[X_{nk}^2\right], \quad s_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2.$$

The aim is to show that the *central limit theorem* (CLT) under the *Lindeberg condition*:

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{|X_{nk}|\geq \epsilon s_n} X_{nk}^2 d\mathbb{P} = 0, \quad \text{for all } \epsilon > 0 \quad \Longrightarrow \quad \frac{S_n}{s_n} \Rightarrow N(0,1). \qquad \text{(CLT)}$$

Carry out your arguments in the following steps:

1. (5 points) Show that

$$\left| e^{itx} - \left(1 + itx - \frac{1}{2}t^2x^2\right)\right| \leq \min\left(|tx|^2, |tx|^3\right).$$

2. (5 points) Show that for all $z_1, \cdots, z_m$, and $w_1, \cdots, w_m \in \mathbb{C}$ such that $|z_i| \leq 1$ and $|w_i| \leq 1$, it holds that

$$\left|\prod_{i=1}^m z_i - \prod_{i=1}^m w_i\right| \leq \sum_{i=1}^m |z_i - w_i|.$$

3. (15 points) Prove the (CLT) using parts 1 and 2.

Theorem 27.2 on p. 359, which states the central limit theorem under the Lindeberg condition. We have covered the proof of this theorem in detail in class. Both the result and its proof are highlights of the course.