2010 Mathematical Statistics (STAT 7600/7610) Preliminary Exam
Ph.D. Program, Department of Mathematics and Statistics, Auburn University
Exam Produced by Statistics Group, Department of Mathematics and Statistics

INSTRUCTIONS FOR 2010 PRELIM:

Exam Time/Date: 4:00pm-6:30pm, Sunday, October 17, 2010

1. It is a closed-book, in-class written exam.
2. Calculator is allowed. No laptop (or equivalent).
3. Show your work to receive full credit. Highlight your final answer.
4. Solve any five problems out of the seven problems.
5. Total points are 50. Each question is worth 10 points.
6. If you work out more than five problems, your score is the sum of five highest points.
7. Time: 150 minutes. (4:00pm–6:30pm, Sunday, October 17, 2010)

1	2	3	4	5	6	7	Total

NOTE: You must show all your work on each problem. You will be graded on correctness, completeness, and style.

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1. Let $X \in \mathcal{R}^p$ denote a real valued random input vector and $Y \in \mathcal{R}$ a real valued random output variable, with joint distribution $\Pr(X, Y)$. We seek a function $f(X)$ for predicting Y given values of the input X. Based on a loss function L(X,f(X)) for penalizing errors in prediction, we would like to find the function that minimizes the expected prediction error defined as $EPE(f) = EL\big(Y, f(X)\big)$. For the following two loss functions find an expression for the function that minimizes the EPE (in each case you must demonstrate that the function does indeed minimize EPE):

   (a) Square error loss, $L\big(Y, f(X)\big) = (Y - f(X))^2$
   (b) Absolute error loss, $L\big(Y, f(X)\big) = |Y - f(X)|$.

2. Prove The Poisson Postulates Theorem: For each $t \geq 0$, let N_t be an integer-valued random variable with the following properties. (Think of N_t as denoting the number of arrivals in the time period from time 0 to time t.)
   (i) $N_0 = 0$ (start with no arrivals)
   (ii) $s < t \Longrightarrow N_s$ and $N_t - N_s$ are independent (arrivals in disjoint time periods are independent)
   (iii) N_s *and* $N_{t+s} - N_t$ are identically distributed (number of arrivals depends only on period length)
   (iv) $\lim_{t\to 0} \frac{P(N_t=1)}{t} = \lambda$ (arrival probability proportional to period length, if length is small)
   (v) $\lim_{t\to 0} \frac{P(N_t>1)}{t} = 0$ (no simultaneous arrivals)
   (vi) If i-v hold, then for any integer n,
   $P(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, i.e., N_t \sim Poisson(\lambda t).$

3. For any two random variables X and Y,
   (a) prove that $E(X) = E\big(E(X|Y)\big)$ *and* $VarX = E\big(Var(X|Y)\big) + Var(E(X|Y))$.
   (b) If $X|Y \sim binomial(Y, p)$ *and* $Y \sim Poisson(\lambda)$, then find EX and Var(X).
   (c) Find the marginal distribution of X given in (b).

4. (a) Prove Basu's Theorem: If T(**X**) is a complete and minimal sufficient statistic, then T(**X**) is independent of every ancillary statistic. (b) use Basu's Theorem to show that $g(X) = \frac{X_n}{X_1+\cdots+X_n}$ *and* $T(X) = X_n + \cdots + X_n$ *are independent, where* $X_1, \ldots, X_n$ *are iid exponentials with parameter* θ.

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5. Let $X_1, \ldots, X_n$ be iid Poisson (λ) and let $\bar{X}$ *and* S^2 denote the sample mean and variance, respectively.
   (a) Prove that $\bar{X}$ is the best unbiased estimator of λ.
   (b) Prove the identity $E(S^2|\bar{X}) = \bar{X}$, and use it to explicitly demonstrate that Var$(S^2) > Var\ (\bar{X})$.
   (c) Using completeness, can a general theorem be formulated for which the identity in part (b) is a special case? If so, show it.

6. Let $X_1, \ldots, X_n$ be iid Poisson(λ), and let λ have a gamma(α,β) distribution, the conjugate family for the Poisson.
   (a) Define conjugate family, in general.
   (b) Find the posterior distribution of λ.
   (c) Find the Bayes estimator of λ and derive its variance.

7. State and Prove the Central Limit Theorem (use any version you want, but state the assumptions that you make).

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