The Prelim Exam in Modern Stochastic Processes, MATH 7810-20

October 4, 2010

Show all of your work. Quote the theorems that you use. Work on each problem 1,2,... on a separate sheet of paper. The passing level is 66.66%.

1. Let M_t be a nonhomogeneous Poisson process on $(0, \infty)$ with the intensity function $\lambda(t) = \dfrac{1}{2\sqrt{t}}$ and signals T_n.

(a) Find a function $\phi : [0, \infty) \to [0, \infty)$ such that $T_n = \phi(S_n)$, where S_n are arrival times of a standard Poisson N_t process with unit intensity.

(b) Find the mean and variance of the number of signals T_n in the interval $[1, 100]$.

(c) Are the times $T_n - T_{n-1}$ between the signals independent random variables? Explain.

2. Consider a Markov chain model for the nursery rhyme:

The eensey-weensey spider went up the water spout.
Down came the rain and washed the spider out.
Out came the sun and dried up all the rain.
Then the eensey-weensey spider went up the spout again.

That is, the spider moves along an infinite discrete water spout represented by the whole numbers 0, 1, 2, ... The spider goes up one step with probability p and is washed down by the rain to level 0 with probability $q = 1 - p$.

Auxiliary tasks:

a) *Write the transition matrix.*
b) *Is the Markov chain irreducible*? Explain.
c) *Is the process periodic or aperiodic*? Explain.

The main objective:

Find the stationary probabilities π_n *when they exist.*
(At the very least write the suitable equations and comment on ways of solving them.)

3. Formulate precisely (with assumptions and theses) the three basic renewal theorems (RT, in short) involving a renewal process $N(t)$ and the renewal function $m(t) = \mathsf{E}\, N(t)$:

- The Basic RT about the limit of $\dfrac{N(t)}{t}$;
- The Blackwell RT about the limit of $m(t+a) - m(t)$;
- The Key RT about the limit of the convolution $h * m$.

The main objectives:

(a) Derive the Basic RT from the Strong Law of Large Numbers for i.i.d. random variables.

(b) Derive the Blackwell RT from the Key RT, and the Key RT from the Blackwell RT.

4. Let X_t denote a standard Brownian Motion.

(a) *Show that X_t is a continuous time martingale.*

The main objectives:

(b) *Show that $Y_t = X_t^2 - at$, $t \geq 0$, is a martingale for some constant a. Find a.*

(c) Fix c. *Show that $G_t = e^{cX_t - bt}$, $t \geq 0$, is a martingale for some constant b. Find $b = b(c)$ as a function of c.*

Let now $X_t = N_t - \lambda\, t$, where N_t is a standard Poisson process with intensity $\lambda > 0$.

(d) *Do all statements (a), (b), (c) still hold?* Explain.

5. Suppose that the assets $A(t)$ at time $t \geq 0$ of some financial institution vary at random, proportionally to values of a standard Brownian motion, $A(t) \stackrel{\mathcal{D}}{=} aB_t$ (as stochastic processes). The institution files for bankruptcy when the assets reach the debt $-b$, where $b > 0$. Let $T = T_b$ denote the waiting time for that event. That is,

$$T_b > t \iff \min_{s \leq t} A(s) > -b.$$

(a) *Illustrate the above relation graphically.*

(b) **The main objective**: Find the probability distribution of T_b,

(c) *Show that* $T_b < \infty$ *with probability 1 but* $\mathsf{E}\, T_b = \infty$.

Remark. It may help to notice that this problem is equivalent to the situation when the institution pulls out of the market once its assets hit the positive level b.

6. Consider a Markov chain with continuous time as a model for the life of an amphibian. That is, the animal stays in water for an exponential time with rate λ before crawling to the land, where it stays for an exponential time with rate μ before returning to water.

(a) **The first main objective**: *Write Kolmogorov's Forward Equations for* $P_{ij}(t)$, where both i, j mean "water" or "land", giving first the definition of these probabilities.

(b) If you happen to see that animal, *what is the probability that it is in water at this moment?*

(c) **The second main objective**: *Solve Kolmogorov's Forward Equations from Part (a).*