STAT 7600/7610 Mathematical Statistics Preliminary Exam

August 14, 2017

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NAME: ______________________________

1. It is a closed-book in-class exam.
2. Calculator is allowed and will be provided by the proctor.
3. The proctor will provide as much blank sheets of paper as you need.
4. Show your work to recieve full credit. Highlight your final answer.
5. Solve any five problems out of seven. Only turn in the five problems you want to be graded or otherwise clearly indicate which five problems you are submitting for evaulation.
6. Turn in your the exam paper (the three typeset pages handed to you) along with your work-sheets stabled to the back. Please place the problems in numerical order and label each at the top.
7. Total points are 50 with 10 points for each problem.
8. Time: 240 minutes. (8:00am-12:00(noon), August 14, 2017).

1	2	3	4	5	6	7	Total

1. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Poisson population with mean λ. Let $\tau(\lambda) = e^{-\lambda} + \lambda e^{-\lambda}$.

   (a) Find the *maximum likelihood estimator* (MLE) for $\tau(\lambda)$, denote the MLE as $\hat{\tau}(\lambda)$. Show all work (derivations) and reasoning.

   (b) Find an unbiased estimator of $\tau(\lambda) = e^{-\lambda} + \lambda e^{-\lambda}$ and denote it as $\hat{\tau}^*(\lambda)$.

   (c) Is there a *best unbiased estimator* of $\tau(\lambda)$? Explain/prove your answer. Note: you do not necessarily have to find an explicit expression for this statistic to prove your answer.

   (d) Compute the asymptotic relative efficiency (ARE) of $\hat{\tau}^*(\lambda)$ with respect to $\hat{\tau}(\lambda)$ and describe its behavior as a function of λ.

2. Suppose $X_1, \ldots, X_n$ are iid exponential(μ, σ), i.e., $(1/\sigma)e^{-(x-\mu)/\sigma} I(x \geq \mu)$, where $-\infty < \mu < \infty$ and $\sigma > 0$.

   (a) Show that $\hat{\theta} = T(\mathbf{X}) = (T_1(X), T_2(X)) = (X_{(1)}, \bar{X} - X_{(1)})$ is the *maximum likelihood estimator* (MLE) for $\boldsymbol{\theta} = (\mu, \sigma)$, where $\bar{X} = \sum X_i/n$ and $X_{(1)} = \min\{X_1, \ldots, X_n\}$. Also, show that $\hat{\boldsymbol{\theta}}$ is *minimally sufficient* for $\boldsymbol{\theta}$, but when σ is known, T_1 is a complete and minimal sufficient.

   (b) Are T_1 and T_2 independent? Prove your answer.

3. Let $X_1, \ldots, X_n$ be iid $n(\theta, \sigma^2)$, both θ and σ unknown. We are interested in testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0.$$

   Recall, the unrestricted MLEs are $\hat{\theta} = \bar{X}$ and $\hat{\sigma}^2 = \sum(X_i - \bar{X})^2/n$. The restricted MLE's are $\hat{\theta}_0 = \theta_0$ and $\hat{\sigma}_0^2 = \sum(X_i - \theta_0)^2/n$.

   (a) Show that the test that rejects H_0 when $|\bar{X} - \theta_0| > t_{n-1,\alpha/2}\sqrt{S^2/n}$ is a test of size α.

   (b) Show that the test in part (a) can be derived as a *likelihood ratio test* (LRT).

   (c) For σ unknown and θ known, find a Wald statistic for testing $H_0 : \sigma = \sigma_0$.

4. Suppose $X_1, X_2, \ldots, X_n$ are a random sample from a $N(\mu, \sigma^2)$, where the mean μ is known and the variance σ^2 is unknown.

   (a) For a given σ, find the exact probability density function (pdf) of $\dfrac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2$ and, if possible, name this distribution. Show work/derivation.

   (b) For the hypotheses $H_0 : \sigma \leq \sigma_0$ versus $H_1 : \sigma > \sigma_0$, find the *uniformly most powerful test.* For a given value of α, the size of the Type I error, show how the value of c (the threshold for the test statistic) is explicity determined.

   (c) Derive a $1 - \alpha$ lower confidence bound for σ.

5. Let $X_1, X_2, \ldots, X_n$ be iid Poisson (λ) and let λ have a gamma(α, β) prior distribution.

   (a) Find the posterior distribution of λ. Show your derivation/work.

   (b) Find the Bayes estimator of λ and compute its variance.

   (c) Find a $(1-\alpha)$ *credible set* for λ, where $0 < \alpha < 1$.

6. Suppose $X_1, X_2, \ldots, X_n$ are iid with a continuous cdf $F_X(\cdot)$. Let $\hat{F}_n(x) = \dfrac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$ denote the *empirical distribution function* (edf).

   (a) Does $\hat{F}(x)$ converge in probability to $F_X(x)$ at each fixed $x \in \mathcal{R}$? Prove your answer.

   (b) Suppose this sample were from an exponential (σ) population, i.e., $f_X(x|\sigma) = \frac{1}{\sigma}e^{-x/\sigma}, x > 0$. Find an expression for MLE of the $F_X(x)$ and denote it as $\hat{F}^*_X(x)$.

   (c) Suppose we took a sample of size $n = 5$ from an exponential(σ) population and the sample observations were 0.9, 0.4, 0.7, 0.2 and 2.0. Compute the MLE, $\hat{F}^*(x)$ derived in (b). Sketch a graph of $\hat{F}^*(x)$ overlaid with the edf $\hat{F}_n(x)$ on the same graph (let the horizontal axis range from 0 to 4).

7. For testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, suppose that $X_1, \ldots, X_n$ are iid $f(x|\theta)$, $\hat{\theta}$ is the MLE of θ, and $f(x|\theta)$ satisfies regularity conditions.

   (a) Show that under H_0, as $n \to \infty$

   $$-2\log\lambda(\mathbf{X}) \to \chi^2_1 \text{ in distribution,}$$

   where χ^2_1 is a chi-square random variable with 1 degree of freedom and $\lambda(\mathbf{X})$ is the likelihood ratio.

   (b) Suppose the random sample was from a Poisson (θ) population. Using the result in (a), derive the expression for the approximate chi-square test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, based on $-2\log\lambda(\mathbf{x})$ for this Poisson sample.