STAT 7600/7610 Statistical Theory Preliminary Exam

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Name:________________________

1. It is a closed-book in-class exam. You are allowed to have formula sheets of two pages and double-sided.
2. A calculator is allowed.
3. The proctor will provide as many blank sheets of paper as you need.
4. Show your work to receive full credit. Highlight your final answer.
5. Turn in your the exam paper (the three typeset pages handed to you) along with your work-sheets stabled to the back.
6. Planned Time: 240 minutes (8:00 am-12:00(noon)).
7. **Five** problems will be graded. Problems 1 and 2 are mandatory. Then, you **must** select **three problems** from Problems 3–6 to submit and grade. The rest problems will not be graded. Indicate your selections in the table.

1	2	3	4	5	6	Total

Gamma function

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx, \quad a > 0.$$

The density, mean, and variance of selected common distributions.

- Normal (μ, σ^2)

$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad E(X) = \mu \quad \text{var}(X) = \sigma^2$$

- Gamma(α, β) (shape-rate parametrization)

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad E(X) = \alpha/\beta \quad \text{var}(X) = \alpha/\beta^2$$

If $X_1 \sim \text{Gamma}(\alpha_1, \beta)$, $X_2 \sim \text{Gamma}(\alpha_2, \beta)$, and X_1 is independent of X_2, then

$$\beta(X_1 + X_2) \sim \text{Gamma}(\alpha_1 + \alpha_2, 1).$$

- χ^2_p

$$f(x) = \frac{1}{\Gamma(p/2) 2^{p/2}} x^{(p/2)-1} e^{-x/2} \quad E(X) = p \quad \text{var}(X) = 2p$$

- Beta(α, β)

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad E(X) = \frac{\alpha}{\alpha+\beta} \quad \text{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

- Poission(λ)

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad E(X) = \lambda \quad \text{var}(X) = \lambda$$

Throughout, the asymptotics whenever mentioned is in terms of the sample size approaching infinity. **Each question is worth 20 points**.

Mandatory Questions

Problem 1: Consider the following model,

$$Y \mid X \sim N\left(\mu, X^{-1}\right), \quad X \sim \text{Gamma}(a/2, b/2),$$

where $\text{var}(Y \mid X) = X^{-1}$ and the density of $\text{Gamma}(a, b)$ is

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, \quad x > 0$$

(a) Find the marginal density of Y.

(b) Compute $E(Y)$ and $\text{var}(Y)$. Hint: $\text{var}(Y) = E(\text{var}(Y \mid X)) + \text{var}[E(Y \mid X)]$.

Problem 2: Let $X_1, \ldots X_n$ be i.i.d. following the **discrete** distribution

$$f(x \mid \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1,$$

where $\theta \in (0, 1)$ is the parameter.

(a) Find the maximum likelihood estimator (MLE) of θ.

(b) Find the asymptotic distribution of the MLE.

(c) Find an approximate 95% confidence interval for θ.

(d) Derive the likelihood ratio test for testing $H_0 : \theta = 1/2$ against $H_a : \theta > 1/2$. State how the critical value can be determined at an asymptotic level α.

Choose Three From Four

Problem 3: Let $X \sim N(\lambda\mu, \Sigma)$, where $\mu \in \mathbb{R}^n$ is a **known** vector and $\Sigma \in \mathbb{R}^{n\times n}$ is a **known** positive definite matrix; $\lambda \in \mathbb{R}$ is unknown parameter.

(a) Find the MLE $\hat{\lambda}$ of λ based on the observation X.

(b) Is the MLE of λ unbiased. Justify your answer.

(c) Find the variance of $\hat{\lambda}$.

(d) Suppose that it is further known that $\mu = (1, 1, \ldots, 1)^T$ and $\Sigma = I_n$ (the identity matrix). Find the (exact) distribution of $\hat{\lambda}$.

(e) Design an (exact) level α test for the hypothesis: $H_0 : \lambda = 1$ v.s. $H_a : \lambda \neq 1$.

Problem 4: Let (Y, X) follows a truncated bivariate normal distribution conditional on $Y > 0$ and $X > 0$ that is, its joint density of (Y, X) is

$$f(y, x) = \frac{\phi(y, x)}{P(Y > 0, X > 0)}, \quad x > 0, y > 0$$

where $\phi(y, x)$ is the joint density of bivariate normal distribution given by

$$\phi(y, x) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2\left(1-\rho^2\right)}\right\}, \quad -1 < \rho < 1.$$

(a) Show that the marginal distribution of Y is not truncated normal.

(b) Show that the conditional distribution of $Y \mid X$ is truncated normal.

Note that the truncated normal distribution $N\left(\mu, \sigma^2, (a, b)\right)$ (restricted to (a, b)) has the density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}\left[\Phi\left(b; \mu, \sigma^2\right) - \Phi\left(a; \mu, \sigma^2\right)\right]} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}, \quad a < x < b, a < b$$

where $\Phi\left(x; \mu, \sigma^2\right)$ is the CDF of $N\left(\mu, \sigma^2\right)$.

Problem 5: Assume X is a random variable with pdf

$$f(x \mid \alpha, \beta) = \frac{1}{\lambda(\alpha, \beta)} x^{\frac{\alpha}{\beta}-1} e^{-\frac{x^2}{2\beta}}, \quad x > 0,$$

where

$$\lambda(\alpha, \beta) = \int_0^\infty x^{\frac{\alpha}{\beta}-1} e^{-\frac{x^2}{2\beta}} dx$$

and $\theta = (\alpha, \beta)$ lies in the parameter space

$$\Theta = \{(\alpha, \beta) : \alpha > 0, \beta > 0\}$$

(a) Is the distribution part of the exponential family distribution? Express the distribution in its canonical form:

$$f(x \mid \eta) = h(x) \exp(\eta^T T(x) - A(\eta)).$$

Provide all components of the expression and the natural parameter space. Note that η and $T(x)$ are possibly vectors.

(b) Find a complete and sufficient statistic (vector) for θ; and justify your answer.

(c) Find the UMVUE (uniformly minimum variance unbiased estimator) for α; and justify your answer. Hint: $E(T(X))$ equals the gradient of $A(\eta)$, and $A(\eta)$ is related to the Gamma function.

Problem 6: Let $X_1, \ldots, X_n$ be i.i.d. with pdf

$$f(x \mid \lambda) = \frac{3}{\lambda}\left(\frac{x}{\lambda}\right)^2 \exp(-(x/\lambda)^3), \quad x > 0,$$

where $\lambda > 0$ is the parameter. It is known that $EX^3 = \lambda^3$.

(a) Find the MLE of λ^3.

(b) Denote the MLE in (a) by $T(X)$. Find the asymptotic distribution of $T(X)$.

(c) Show that this family of distributions has the monotone likelihood ratio property with respect to $T = T(X_1, \ldots, X_n)$.

(d) For the statistical model described, given an example of a hypothesis testing problem $H_0 : \lambda \in \Theta_0$, $H_a : \lambda \in \Theta_1$, where Θ_0 and Θ_1 are two subsets of $(0, \infty)$ that satisfy $\Theta_0 \cap \Theta_1 = \emptyset$, $\Theta_0 \cup \Theta_1 = (0, \infty)$, that admits a *uniformly most powerful test* (UMP) of any size $\alpha \in (0, 1)$. Justify your answer. Find the UMP and the critical value at the asymptotic level α.

Solution

Problem 1: (a) The joint density of Y and X is

$$\begin{aligned} f(y,x) &= f(y \mid x) f(x) \\ &= \frac{1}{\sqrt{2\pi x^{-1}}} \exp\left\{-\frac{x(y-\mu)^2}{2}\right\} \frac{(b/2)^{a/2}}{\Gamma(a/2)} x^{a/2-1} e^{-bx/2} \\ &= \frac{1}{\sqrt{2\pi}} \frac{(b/2)^{a/2}}{\Gamma(a/2)} x^{(a+1)/2-1} \exp\left\{-\frac{x\left[b+(y-\mu)^2\right]}{2}\right\} \end{aligned}$$

Hence, the marginal density of Y is

$$\begin{aligned} f(y) &= \int_0^x f(y,x)dx \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{(b/2)^{a/2}}{\Gamma(a/2)} x^{(a+1)/2-1} \exp\left\{-\frac{x\left[b+(y-\mu)^2\right]}{2}\right\} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{(b/2)^{a/2}}{\Gamma(a/2)} \Gamma((a+1)/2) \left[\frac{b+(y-\mu)^2}{2}\right]^{-(a+1)/2} \\ &= \frac{\Gamma((a+1)/2)}{\Gamma(a/2)\sqrt{b\pi}} \left[1+\frac{(y-\mu)^2}{b}\right]^{-(a+1)/2} \end{aligned}$$

where we use the property

$$\int_0^\infty \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} dx = 1 \text{ or } \int_0^\infty x^{a-1} e^{-bx} dx = \Gamma(a) b^{-a}$$

(b) It is difficult to evaluate the moments of Y from its density directly.

$$\begin{aligned} E(Y) &= E[E(Y \mid X)] = \mu \\ \operatorname{var}(Y) &= E[\operatorname{var}(Y \mid X)] + \operatorname{var}[E(Y \mid X)] = E\left(X^{-1}\right) = \frac{b}{a-2} \end{aligned}$$

where

$$E\left(X^{-1}\right) = \int_0^\infty x^{-1} \frac{(b/2)^{a/2}}{\Gamma(a/2)} x^{a/2-1} e^{-bx/2} dx = \frac{(b/2)^{a/2}}{\Gamma(a/2)} \frac{\Gamma(a/2-1)}{(b/2)^{(a/2-1)}} = \frac{b}{a-2}$$

Problem 2: (a) The likelihood function is

$$L(\theta \mid X) = \left(\frac{\theta}{2}\right)^s (1-\theta)^{n-s},$$

where $s = \sum_{j=1}^n |X_j|$. The log-likelihood is

$$\ell(\theta \mid X) \propto s\log(\theta) + (n-s)\log(1-\theta).$$

$$\frac{d\ell(\theta)}{d\theta} = \frac{s}{\theta} - \frac{(n-s)}{1-\theta} = 0.$$

The MLE is then

$$\hat{\theta} = s/n = \frac{1}{n}\sum_{j=1}^n |X_j|.$$

(b) The mean and variance of the distribution of $|X|$ is

$$E|X| = |-1| \times f(-1) + 1 \times f(1) = \frac{\theta}{2} + \frac{\theta}{2} = \theta.$$

$$\text{var}(|X|) = EX^2 - (E|X|)^2 = \theta - \theta^2.$$

Then, by CLT

$$\sqrt{n}(\hat{\theta} - \theta)/\sqrt{\theta(1-\theta)} \longrightarrow N(0,1).$$

(c) Using the asymptotic distribution of $\hat{\theta}$, we get a confidence interval for θ being

$$-z_{1-\alpha/2}/\sqrt{n} \le \frac{\hat{\theta} - \theta}{\sqrt{\theta(1-\theta)}} \le z_{1-\alpha/2}/\sqrt{n}.$$

Or,

$$-z_{1-\alpha/2}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}} \le \theta \le z_{1-\alpha/2}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}.$$

(d) The restricted likelihood under the null hypothesis $\theta = 1/2$ is

$$L(1/2) = (1/4)^s(1/2)^{n-s} = (1/2)^{2s+n-s} = (1/2)^{n+s} = (1/2)^{n(1+\hat{\theta})}.$$

The unrestricted likelihood is

$$L(\hat{\theta}) = (\hat{\theta}/2)^{n\hat{\theta}}(1-\hat{\theta})^{n(1-\hat{\theta})}.$$

The scaled likelihood ratio is

$$\Lambda = -2\log\frac{L(1/2)}{L(\hat\theta)} = -2\left(-n\log 2 - n\hat\theta\log\hat\theta - n(1-\hat\theta)\log(1-\hat\theta)\right)$$

The null hypothesis is rejected at asymptotic level $1-\alpha$ if

$$\Lambda > \chi_1^2(1-\alpha).$$

Problem 3: (a) The log-likelihood is such that

$$\ell(\lambda) \propto -\frac{1}{2}X^T\Sigma^{-1}X + \lambda\mu^T\Sigma^{-1}X - \frac{1}{2}\lambda^2\mu^T\Sigma^{-1}\mu.$$

$$\frac{d\ell(\lambda)}{d\lambda} = \mu^T\Sigma^{-1}X - \lambda\mu^T\Sigma^{-1}\mu.$$

The MLE is

$$\hat\lambda = \frac{\mu^T\Sigma^{-1}X}{\mu^T\Sigma^{-1}\mu}.$$

(b)

$$E(\hat\lambda) = E\frac{\mu^T\Sigma^{-1}X}{\mu^T\Sigma^{-1}\mu} = \frac{\mu^T\Sigma^{-1}(\lambda\mu)}{\mu^T\Sigma^{-1}\mu} = \lambda.$$

Therefore, $\hat\lambda$ is unbiased.

(c)

$$E(\hat\lambda-\lambda)^2 = \frac{1}{(\mu^T\Sigma^{-1}\mu)^2}\mu^T\Sigma^{-1}E(X-\lambda\mu)(X-\lambda\mu)^T\Sigma^{-1}\mu = (\mu^T\Sigma^{-1}\mu)^{-1}.$$

(d) Under the specified value of μ and Σ, we conclude that the entries of X are i.i.d. $N(\lambda, 1)$.

$$\hat\lambda = \frac{1}{n}\mu^TX = \bar X,$$

where $\bar X$ is the average of entries in X. It then follows that

$$(\hat\lambda \sim N(\lambda, 1/n).$$

(e) The two-sided z-test rejects the null hypothesis at level α if

$$|\bar X - 1| \geq z_{1-\alpha/2}.$$

Problem 4: (a) The marginal density of Y is

$$
\begin{aligned}
f(y) &= \int_0^\infty \frac{\phi(y,x)}{P(Y>0,X>0)}dx \\
&= \frac{1}{P(Y>0,X>0)}\frac{1}{2\pi\sqrt{1-\rho^2}}\int_0^\infty \exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}dx \\
&= \frac{1}{P(Y>0,X>0)}\frac{1}{2\pi\sqrt{1-\rho^2}}\int_0^\infty \exp\left\{-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right\}\exp\left\{-\frac{y^2}{2}\right\}dx \\
&= \frac{1-\Phi\left(0;\rho y,(1-\rho^2)\right)}{P(Y>0,X>0)}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{2}\right\}
\end{aligned}
$$

This density is not the density of a truncated normal distribution.

(b) The conditional density of $Y \mid X$ is

$$
\begin{aligned}
f(y\mid x) =& \frac{f(y,x)}{f(x)} \\
=& \frac{1}{P(Y>0,X>0)}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\} \\
&\cdot \frac{P(Y>0,X>0)}{1-\Phi\left(0;\rho x,(1-\rho^2)\right)}(\sqrt{2\pi})\exp\left\{\frac{x^2}{2}\right\} \\
=& \frac{1}{1-\Phi\left(0;\rho x,(1-\rho^2)\right)}\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left\{-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right\}
\end{aligned}
$$

which is exactly the density of $N\left(\rho x,(1-\rho^2),(0,\infty)\right)$.

Problem 5: (a) The distribution is an exponential family distribution

$$
f(x\mid\alpha,\beta) = \frac{1}{x}\exp\left(-\frac{1}{2\beta}x^2+\frac{\alpha}{\beta}\log x-\log\lambda(\alpha,\beta)\right).
$$

Let

$$
\begin{aligned}
&h(x) = 1/x, \\
&T(x) = (T_1(x),T_2(x)) = (x^2,\log x), \\
&\eta = (\eta_1,\eta_2) = \Big(-\frac{1}{2\beta},\frac{\alpha}{\beta}\Big), \\
&A(\eta) = \log(\lambda(-\eta_2/(2\eta_1),-1/(2\eta_1))) = \log\int_0^\infty x^{\eta_2-1}e^{\eta_1x^2}dx = \log\left(\frac{1}{2}(-\eta_1)^{-\eta_2/2}\Gamma(\eta_2/2)\right) \\
&\quad = -\log(2)-(\eta_2/2)\log(-\eta_1)+\log\Gamma(\eta_2/2)
\end{aligned}
$$

The canonical form is

$$f(x \mid \eta_1, \eta_2) = h(x) \exp\left(\eta_1 T_1(x) + \eta_2 T_2(x) - A(\eta_1, \eta_2)\right).$$

The natural parameter space is

$$\Xi = \{(\eta_1, \eta_2) \mid |A(\eta)| < \infty\} = \{(\eta_1, \eta_2) \mid \eta_1 < 0, \eta_2 > 0\}.$$

(2) The natural sufficient statistic is $T(X) = (X^2, \log X)$. Since the family is of full rank. Therefore, $T(X)$ is complete.

(3) Consider

$$\frac{\partial A}{\partial \eta_1} = \frac{-\eta_2/2}{\eta_1} = -\frac{\eta_2}{2\eta_1} = \alpha.$$

Therefore,

$$EX^2 = \frac{\partial A}{\partial \eta_1} = \alpha.$$

Since $T(X)$ is sufficient and complete, we have X^2 is the UMVUE for α.

Problem 6: (a) The log-likelihood is

$$\ell(\lambda) = \sum_{j=1}^{n} \log X_j^2 - 3n \log \lambda - \frac{1}{\lambda^3} \sum_{j=1}^{n} X_j^3.$$

The MLE of λ is

$$\hat{\lambda} = (\frac{1}{n} \sum_{j=1}^{n} X_j^3)^{1/2}.$$

The MLE of λ^3 is

$$T(X) = \hat{\lambda}^3 = \frac{1}{n} \sum_{j=1}^{n} X_j^3.$$

(b) The Fisher information is

$$I_n(\lambda) = -E \frac{\partial^2}{\partial \lambda^2} \ell(\lambda) = -\frac{3n}{\lambda^2} + \frac{12n}{\lambda^2} = \frac{9n}{\lambda^2}.$$

Therefore, the CLT of MLE yields

$$\sqrt{n}(\hat{\lambda} - \lambda) \Longrightarrow N(0, I_1^{-1}(\lambda)).$$

That is,

$$\sqrt{n}(\hat{\lambda} - \lambda) \Longrightarrow N(0, \lambda^2/9).$$

By the delta-method,

$$\sqrt{n}(\hat{\lambda}^3 - \lambda^3) \Longrightarrow N\Big(0, (3\lambda^2)^2\lambda^2/9\Big).$$

That is,

$$\sqrt{n}(T(X) - \lambda^3) \Longrightarrow N(0, \lambda^6).$$

(c) Considet two values of λ, say $\lambda_1 < \lambda_2$.

$$f(x_1, \ldots, x_n \mid \lambda_2)/f(x_1, \ldots, x_n \mid \lambda_1) = \frac{\lambda_1^3}{\lambda_2^3} \exp\Big((1/\lambda_1^3 - 1/\lambda_2^3) \sum_{j=1}^{n} x^3\Big)$$

The ratio is an increasing function of the statistic $T(X) = n^{-1} \sum_{j=1}^{n} X_j^3$. Therefore, this family has the monotone likelihood ratio property with respect to T.

For a one-sided hypothesis, such as

$$H_0 : \lambda < 1 \quad \text{against} \quad H_a : \lambda > 1,$$

The likelihood ratio test is the UMP. In this case, the likelihood ratio test is equivalent to reject the null hypothesis if $T(X)$ is large. Namely,

$$T(X) > c.$$

Here, the critical value is selected so that the asymptotic type I error rate is α. Using Part (b), we have

$$\sqrt{n}\frac{T(X) - \lambda^3}{T(X)} \Longrightarrow N(0, 1).$$

Then, the asymptotic size α likelihood ratio test for the hypothesis $H_0 : \lambda < 1$ v.s. $H_a : \lambda > 1$ rejects the null hypothesis if

$$\sqrt{n}(1 - 1/T(X)) > z_{1-\alpha}.$$

That is,

$$T(X) > \frac{1}{1 - (1/\sqrt{n})z_{1-\alpha}}.$$