The weighted Weierstrass Theorem for continuous functions defined on $[0, \infty)$ or on $(-\infty, \infty)$, proved using Bernstein-Chlodovski operators

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The Weierstrass Approximation Theorem is one of the fundamental results upon which several intersecting areas of mathematics are based. It says that any function which is continuous on a closed interval can be approximated within arbitrary accuracy by an algebraic polynomial. The article [2] of S. Bernstein was not the first proof of the Weierstrass Approximation Theorem. But Bernstein's proof is one of the best known proofs of the result, found nowadays in a multitude of textbooks, articles, and research monographs. Bernstein's proof introduced a sequence of approximation operators which were based upon minor adaptation of the Binomial Theorem, and then demonstrated that those operators must converge uniformly to the function to be approximated.

This talk will deal with the question of extending the Weierstrass Approximation Theorem to apply to continuous functions whose domain is an unbounded interval, either $[0, \infty)$ or $(-\infty, \infty)$. The norm of such a continuous function f will be defined as $\|f\|_W = \sup \|W(x)f(x)\|$, this supremum being taken over the donain of f. It will be further assumed that $W(x)$ is an exponentially decaying weight, of the form $W(x) = e^{-x^\alpha}$ if the domain is $[0, \infty)$, or $W(x) = e^{-|x|^\alpha}$ if the domain is $(-\infty, \infty)$. It is further assumed that $W(x)f(x) \to 0$ as $x \to \infty$, or, respectively, as $|x| \to \infty$.

The proof presented here will extend Bernstein's classical proof of the Weierstrass Theorem, establishing the appropriate weighted version of the Weierstrass Theorem for the functions described above. Just as Bernstein did not give the first proof of the Weierstrass Theorem for functions in $C[a, b]$, this proof is not the first proof of the Weierstrass Theorem for exponentially weighted functions defined on unbounded intervals. However, those who pioneered the contemporary theory regarding weighted approximation on unbounded intervals rejected out of hand the approach which will be described here, apparently believing it to be impossible. Therefore, they turned instead to the development of the theory of weighted orthogonal polynomials. The proof of the Weierstrass Theorem presented here will use none of their results, but will independently establish the Weierstrass theorem with a new proof which is simple, basic in character, completely self-contained and autonomous. As a result, a major portion of the existing theory can be developed by completely alternative means.

The first attempts to approximate functions on unbounded intervals with the Bernstein's methods were Chlodovski's extensions [3] of the classical Bernstein operators [2]. In this, he was in some ways quite successful. However, his results addressed pointwise convergence, and convergence on compacta. Chlodowski did not deal in any way with weighted approximation. Nevertheless, his work does provide a starting point here.

The first results which actually deal with the topic in this talk were the articles Kilgore [4] and [5]. These results and their proofs will be presented here.

It will also be seen that these two articles do not entirely resolve the question, as the methods presented in them do not apply for all of the values of the exponent α for which the result ought to hold. Some very old results, found in the articles [1] and [6], are relevant here, as they establish what the permissible values of α actually ought to be.

Finally, it will be seen that there are simple methods which can be used in order to get around the problems which cause the results in [4] and [5] to be incomplete.

References

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[4] T. Kilgore, On a constructive proof of the Weierstrass Theorem with a weight function on unbounded intervals, *Mediterr. J. Math.*, **14** 6, December 2017, article number 217.

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