RESEARCH ARTICLE

IONIZATION OF HYDROGEN ATOMS BY A LOW-FREQUENCY LASER FIELD OF AN ARBITRARY STRENGTH

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Abstract: We present a further development of an analytical model for the tunneling ionization of atoms by a low-frequency laser field of an arbitrary strength, including the strong-field region. The model uses a very accurate approximation of the true ionization barrier/potential in the Schrödinger equation by the effective parabolic barrier/potential-based on the algorithm suggested by Miller and Good and later employed by Kulyagin and Taranukhin (KT) for calculating the ionization rate W(F) of hydrogen atoms from the ground state. We point out and eliminate a number of principal errors made by KT and calculate W(F) much more accurately. We demonstrate that the dependence of the ionization rate on the laser field is *monotonic* and does not show any effect of the stabilization ("local ionization suppression") claimed by KT. Our results for W(F) are in a good agreement with the results of quantum fully-numerical simulations. The analytical method, further developed in the present paper, can be extended without difficulty to calculations of the tunneling ionization for a number of other quantum systems.

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1. INTRODUCTION

Ionization of atoms/ions in general and of hydrogen atoms in particular by a strong Oscillatory Electric Field (OEF) of the laser or synchrotron radiation has been studied for many years (see, e.g., reviews [1, 2]). In the situation where $\hbar\omega << E_i$ and simultaneously $F << F_a$, the character of the ionization is controlled by the parameter [3]

$$\gamma = \omega (2E_i)^{1/2}/F. \qquad \ldots (1)$$

Here ω and F are the frequency and the characteristic strength of the OEF, respectively; E_i—the binding energy of the outer atomic electron; F_a—the atomic electric field. In this paper, we use atomic units, except where specified to the contrary.

The case of $\gamma >> 1$ corresponds to the multiphoton ionization of atoms, while in the opposite case of $\gamma << 1$ there occurs the tunneling ionization of atoms. In the latter case the ionization rate for hydrogenlike atoms/ions is given by the following expression:

$$W(F) = 4\,[2U(F)]^{5/2}\,F^{-1} \exp\{-(2/3)\,[2U(F)]^{3/2}/F\}, \qquad \ldots (2)$$

where $U(F)$ is the absolute value of the energy of the atomic level that depends on the OEF strength F due to the Stark effect. First observations of the tunneling ionization were published in [4–6].

However, Eq. (2) is valid only when the probability of the tunneling ionization is very small, which is the case for high and/or wide potential barriers. In the case of strong fields F, for which Eq. (2) becomes invalid, there are no exact analytical solutions of the Schrödinger equation for hydrogen atoms. For this case, one of the ideas for approximate solutions is to substitute the true potential by an effective potential, for which there would be an "exact" analytical solution of the Schrödinger equation (see, e.g., paper [7] by Miller and Good).

Such an approach was used by Kulyagin and Taranukhin (hereafter, KT) in [8]. In that paper, the true

potential barrier for hydrogen atoms was substituted by an effective parabolic barrier. The substitution allowed using the well-known exact analytical result for the transmission coefficient of the parabolic barrier (see, e.g., [9], Sect. 50).

The most important result by KT was a possibility of a *stabilization* of the atoms with respect to the ionization (they called it "local ionization suppression"). Specifically, they found that such stabilization should occur at electric fields about 0.2 . This result is of a great interest from the experimental point of view because previously a stabilization of atoms with respect to the ionization was predicted only at the limit of the multiphoton ionization [10–12] (rather than in the tunneling regime dealt with by KT).

However, our analysis of [8] revealed a number of principal errors made by KT (specified in the next sections of our paper). Therefore, there remained an open question whether or not the main result by KT—the possibility of the stabilization in the tunneling regime—would hold after correcting the errors made by KT.

For addressing this question, it was necessary to perform detailed analytical calculations of the tunneling ionization of hydrogen atoms. In this paper, we performed such calculations in spirit of the approach from [8], but without the errors made by KT. Our results for the dependence of the ionization rate of hydrogen atoms on the field strength (on the amplitude of the OEF) differ significantly from the corresponding KT-results. Our most important finding is that there is no stabilization in the tunneling regime of the ionization.

2. CALCULATION OF THE TRANSMISSION COEFFICIENT FOR THE IONIZATION BARRIER OF HYDROGEN ATOMS

We consider the ionization of hydrogen atoms from the ground state in the situation where the frequency and the amplitude of the OEF correspond to $\gamma << 1$ (see Eq. (1))—just as in KT-paper. Let a hydrogen atom be subjected to an electric field F. Then by using the parabolic coordinates one can obtain from the Schrödinger equation the following differential equation for the "η-part" of the wave function:

$$d^2\chi/d\eta^2 + [E(F)/2 + \beta_2(F)/\eta + F\eta/4 - (m^2 - 1)/(4\eta^2)]\,\chi = 0. \quad \ldots (3)$$

Here the parabolic coordinate $\eta = r - z$, where r is the absolute value of the radius-vector, the Cartesian coordinate z is chosen along the laser field **F**; m is the magnetic quantum number; $E(F)$ is the energy level of the bound state of the hydrogen atom ($E(F) < 0$), and $\beta_2(F)$ is the separation constant. Equation (3) has the form of the one-dimensional Schrödinger equation, where the role of the total energy is played by $E(F)/4$, while the role of the potential energy is played by the following function:

$$U_2(\eta) = -\beta_2/(2\eta) - F\eta/8 + (m^2 - 1)/(8\eta^2). \quad \ldots (4)$$

In the present paper, we consider the ionization of hydrogen atoms from the ground state—just as in KT-paper. In this case we have $m = 0$, so that Eq. (4) takes the form:

$$U_2(\eta, F) = -\beta_2(F)/(2\eta) - F\eta/8 - 1/(8\eta^2). \quad \ldots (5)$$

We note that in the KT-paper the last term in $U_2(\eta)$ was erroneously omitted.

The analytical method chosen by KT for calculating the transmission coefficient through the ionization barrier—the method being actually due to Miller and Good [7]—consists of the following 4 steps presented below in the general form.

Step 1: For a given *arbitrary* barrier/potential $U_2(\eta, F)$ and for some arbitrarily chosen effective energy E/4, to find analytically the classical turning points $\eta_1(F, E)$ and $\eta_2(F, E)$.

Step 2: To represent the actual barrier as some *effective* barrier in the parabolic approximation:

$$U_2(\eta, F) \approx U_P(\eta, F) = -U_{0P} - (k/2)\,[\eta - (\eta_1 + \eta_2)/2]^2, \quad \ldots (6)$$

where U_{0P} is the top of the parabolic barrier (*yet unknown, to be determined at Step 3*) and k is a parabolic profile curvature:

$$k(U_{0P}) = 8\,(-E/4 - U_{0P})/(\eta_2 - \eta_1)^2. \quad \ldots (7)$$

Step 3: *(This is the central point of the method):* To find the parameter U_{0P} by solving *analytically* the following equation:

$$\int_{\eta_1(F,E)}^{\eta_2(F,E)} d\eta \left[-E(F)/4 + U_2(\eta, F)\right]^{1/2} = \int_{\eta_1(F,E)}^{\eta_2(F,E)} d\eta \left[-E(F)/4 + U_P(\eta, F)\right]^{1/2}, \quad \ldots (8)$$

where $U_2(\eta, F)$ in the left side is the *actual* (non-approximated) given potential. It is worth noting that in the quasiclassical limit, integrals from Eq. (8) control the transmission coefficient. After substituting $U_P(\eta, F)$ from (6) into the right side of Eq. (8), a straightforward calculation of the integral in the right side of Eq. (8) yields:

$$\int_{\eta_1(F,E)}^{\eta_2(F,E)} d\eta \left[-E(F)/4 + U_P(\eta, F)\right]^{1/2} = \pi(\eta_2 - \eta_1)\left(-E/4 - U_{0P}\right)^{1/2}/4. \quad \ldots (9)$$

Step 4: To substitute the obtained values of U_{0P} and k (U_{0P}) in the exact analytical solution for the transmission coefficient for the parabolic barrier. It is important to emphasize that the analytical results obtained by this method are *valid for both under-the-barrier ionization and above-the-barrier ionization*.

Now we proceed in accordance to the above algorithm. The classical turning points are the solutions of the equation:

$$U_2(\eta, F) - E(F)/4 = 0. \quad \ldots (10)$$

Following the notations of KT-paper (used also above in Eq. (2)), we denote the absolute value the hydrogen energy level as:

$$U(F) = -E(F); \quad \ldots (11)$$

also from now on for brevity, we will skip the argument F of the functions U, U_2, and β_2 in most expressions. Then Eq. (10) can be explicitly re-written as:

$$U_2(\eta) + U/4 = -[F/(8\eta^2)](\eta^3 - 2U\eta^2/F + 4\beta_2\eta/F + 1/F) = [F/(8\eta^2)](b - \eta)(\eta - a)(\eta + c) = 0. \quad \ldots (12)$$

This is a cubic equation with respect to η. It has one negative (real) root denoted as $-c(F)$, so that $c(F) > 0$. The two other roots $b(F)$ and $a(F)$ either are real and positive, or are complex numbers (mutually conjugated) having a positive real part — whether these roots are real or complex depends on the value of F. For definiteness, we assumed that the real part of $b(F)$ is greater or equal than the real part of $a(F)$.

We note that the KT-equation analogous to Eq. (11) was simpler: it was a quadratic equation. This is because they omitted the term $-1/(8\eta^2)$ in the effective potential $U_2(\eta)$.

We denote the left side of Eq. (8) by $J(F)$:

$$J(F) = (F/8)^{1/2}\,\mathrm{Int}(F),$$

$$\mathrm{Int}(F) = \int_a^b (dx/x)\left[(b - x)(x - a)(x + c)\right]^{1/2} \quad \ldots (13)$$

For calculating Int (F), we introduce the following notations:

$$d = [(a + c)/(b + c)]^{1/2}, \quad j = [a/(b + c)]^{1/2}. \quad \ldots (14)$$

Using the software "Mathematica" for analytical calculations, we obtain:

$$\mathrm{Int} = -i\,(2/3)\,(a + c)^{-1/2}\{[b(a + c) + (a^2 - c^2)][\mathbf{E}(1/d^2) - \mathbf{E}(\arcsin d, 1/d^2)] - [b(2c - a) + a(a + c)][\mathbf{K}(1/d^2) - \mathbf{F}(\arcsin d, 1/d^2)] - 3ab\,[\Pi(1/(d^2 - j^2), 1/d^2) - \Pi(1/(d^2 - j^2), \arcsin d, 1/d^2)]\} \quad \ldots (15)$$

Here $\mathbf{F}(\varphi, m)$, $\mathbf{E}(\varphi, m)$, and $\Pi(n, \varphi, m)$ are elliptic integrals of the first, second, and third kinds, respectively; $\mathbf{K}(m)$, $\mathbf{E}(m)$, and $\Pi(n, m)$ are the complete elliptic integrals of the first, second, and third kinds, respectively.

We make two comments in passing. First, there are alternative expressions for some parts of Eq. (15):

$$\mathbf{E}(1/d^2) - \mathbf{E}(\arcsin d, 1/d^2) = i\,(\pi/4)\,\text{Meijer}\,G[\{\{\}, \{1, 1\}\}, \{\{-1/2, 1/2\}, \{\}\}, d^2], \quad \ldots (16)$$

$$\mathbf{K}(1/d^2) - \mathbf{F}(\arcsin d, 1/d^2) = -i\,d\,\mathbf{K}(1 - d^2), \quad \ldots (17)$$

where Meijer $G[\{\{\}, \{1, 1\}\}, \{\{-1/2, 1/2\}, \{\}\}, d^2]$ is the Meijer G function. Equations (16) and (17) show in particular that the left sides of them are purely imaginary. The expression $[\Pi(1/(d^2 - k^2), 1/d^2) - \Pi(1/(d^2 - k^2), \arcsin d, 1/d^2)]$ is also purely imaginary. Thus, Int in Eq. (15) is real, as it should be.

Second, in the case of the potential (5) without the term $-1/(8\eta^2)$, i.e., for the potential erroneously truncated by KT, the left side of Eq. (8) can be reproduced from our result (15) by setting $c = 0$. Also a direct calculation using the "Mathematica" yields for this case (i.e., for KT-case)

$$\int_a^b dx \left[(b-x)(x-a)/x\right]^{1/2}$$

$$= (2ba^{1/2}/3)\ \{[(a+b)/(ab)^{1/2}]\ \mathbf{E}\ (1-a/b) - 2i\ [\mathbf{K}\ (a/b) - \mathbf{F}\ (\arcsin\ (a/b)^{1/2},\ b/a)]\}.$$

The exact expression for the transmission coefficient $D\ (F)$ of the parabolic barrier is (see, e.g., [9], Sect. 50)

$$D = 1/[1 + \exp\ (-2\pi\alpha)], \qquad \ldots (18)$$

where

$$\alpha\ (F) = -(b-a)\ [(U/4 - U_{0p})/8]^{1/2} \qquad \ldots (19)$$

The derivation of Eq. (19) is given in Appendix A, where we corrected another error by KT: their result for $\alpha\ (F)$ was erroneous. By combining Eqs (9), (18), and (19), we obtain the following transmission coefficient for the ionization barrier of hydrogen atoms in the ground state:

$$D\ (F) = 1/\{1 + \exp\ [2^{3/2}\ J\ (F)]\} = 1/\{1 + \exp\ [F^{1/2}\ \mathrm{Int}\ (F)]\}, \qquad \ldots (20)$$

where $J\ (F)$ is given by Eq. (15).

3. CALCULATION OF THE IONIZATION RATE FOR HYDROGEN ATOMS IN THE GROUND STATE

The ionization rate $W\ (F)$ can be represented in the form

$$W\ (F) = S\ (F)\ D\ (F), \qquad \ldots (21)$$

where $S\ (F)$ is the input flow probability. Following the logic of the KT-paper, the function $S\ (F)$ can be determined by the transition to the weak field limit ($F << F_a$) as follows:

The weak field limit of Int (F) — denoted $\mathrm{Int}_{as}\ (F)$ — can be obtained in the form (see Appendix B):

$$\mathrm{Int}_{as}\ (F) = (2/3)\ (2U/F)^{3/2} - (4/3)\ (2/F)^{1/2}\ (1/2 + \beta_2^2/U)^{1/2}\ \ln\ \{[8U^2/F - 12\beta_2 + 4\ (\beta_2^2 + U/2)^{1/2}]/(\beta_2^2 + U/2)^{1/2}\} + (2/F)^{1/2}\ [(1/3 + \ln 2)\ (1/2 + \beta_2^2/U)^{1/2} - (1 + \ln 2)\ \beta_2]. \qquad \ldots (22)$$

Then we find the weak field limit of $J\ (F)$ — denoted $J_{as}\ (F)$ — by multiplying (22) by $(F/8)^{1/2}$ (see (13)):

$$J_{as}\ (F) = 2U^{3/2}/(3F) - (2/3)\ (1/2 + \beta_2^2/U)^{1/2}\ \ln\ \{[8U^2/F - 12\ \beta_2 + 4\ (\beta_2^2 + U/2)^{1/2}]/(\beta_2^2 + U/2)^{1/2}\} + (1/2)\ [(1/3 + \ln 2)\ (1/2 + \beta_2^2/U)^{1/2} - (1 + \ln 2)\ \beta_2]. \qquad \ldots (23)$$

Substituting (23) in (20) and using Eq. (2) for the ionization rate $W\ (F)$ in the weak field limit, we find the input flow probability:

$$S\ (F) = [4\ (2U)^{5/2}/F]\ \exp\ \{(1/2)\ [(1/3 + \ln 2)\ (1/2 + \beta_2^2/U)^{1/2} - (1 + \ln 2)\ \beta_2]\}\ [(\beta_2^2 + U/2)^{1/2}/[8U^2/F - 12\ \beta_2 + 4\ (\beta_2^2 + U/2)^{1/2}]]^{\gamma},$$
$$\gamma = (4/3)\ (1 + 2\ \beta_2^2/U)^{1/2}. \qquad \ldots (24)$$

Thus, *both for weak and strong fields*, the tunneling ionization rate $W\ (F)$ can be presented in the form:

$$W\ (F) = S\ (F)\ D\ [J\ (F)],$$
$$J\ (F) = (F/8)^{1/2}\ \mathrm{Int}\ (F), \qquad \ldots (25)$$

where Int (F) is given by our general analytical result (15).

4. COMPARISON WITH PREVIOUS ANALYTICAL AND FULLY-NUMERICAL CALCULATIONS

For determining the roots $a\ (F)$, $b\ (F)$, and $-c\ (F)$ of the cubic Eq. (12), one needs specific results for both the absolute value of the energy $U\ (F)$ and the separation constant $\beta_2\ (F)$. For $U\ (F)$ we used the results of quantum fully-numerical calculations by Farrelly and Reinhardt [13]. Then we obtained $\beta_2\ (F)$ from analytical formulas of Kondratovich and Ostrovsky [14]: $\beta_2\ (F)$ is expressed through the parameter $Z_1\ (F)$ from [14] as $\beta_2\ (F) = 1 - Z_1\ (F)$.

The comparison of our results for the ionization rate W with the KT-results is presented in Figs 1 and 2: Fig. 1 shows W versus F, while Fig. 2 shows W versus the laser power density P. It is seen that our results are significantly different from the KT-results. Namely, the dependence of W on F (or W on P) is actually *monotonic* and does not show any effect of the stabilization ("local ionization suppression") claimed by KT. The difference is due to the principal errors made by KT, which we have pointed out and eliminated in the present work.

It is interesting to note that the omission of the term $-1/(8\eta^2)$ by KT from the potential $U_2\ (F)$ is not the reason why they found (incorrectly) the "local ionization suppression". We performed also the calculations without the term $-1/(8\eta^2)$ in the potential $U_2\ (F)$ from (5): the result was still a monotonic dependence of W on F. Further, our analysis showed that in the KT-result for $W\ (F)$, the peak at about $F = 0.15$ corresponds to the boundary between under-the-barrier ionization at $F < 0.15$ and above-the-barrier ionization at $F > 0.15$. In the

equation determining the turning points η_1 and η_2 (which is a quadratic equation if the term $-1/(8\eta^2)$ is omitted from the potential $U_2(F)$), the transition from under-the-barrier ionization to above-the-barrier ionization corresponds to the transition from two real roots η_1 and η_2 to two complex roots η_1 and η_2. Even if the term $-1/(8\eta^2)$ is omitted, the properly calculated dependence $W(F)$ remains monotonic at such transition — in distinction to the KT-results. So, it seems that the reason why KT found the decrease of $W(F)$ at the interval $F = 0.15 - 0.20$ is that they did not adequately account for the transition from under-the-barrier ionization to above-the-barrier ionization.

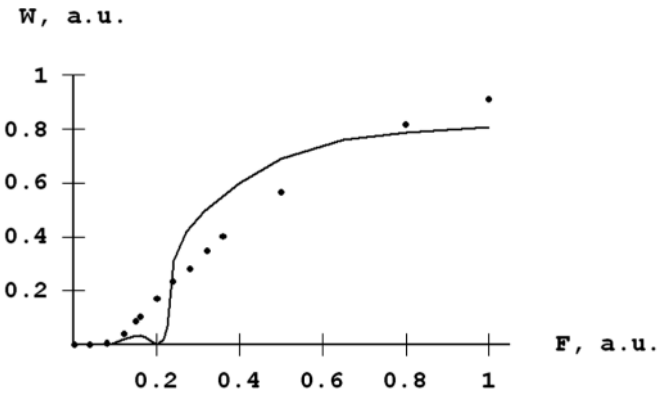

Fig. 1: Dependence of the ionization rate W of hydrogen atoms in the ground state on the laser field F (W and F are in atomic units): dots – our analytical results, solid line – analytical results by Kulyagin and Taranukhin [8]

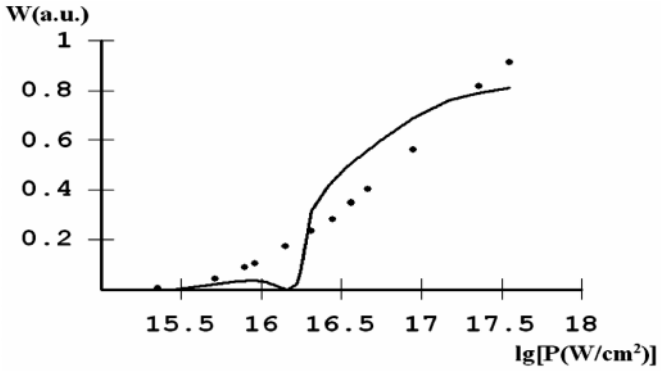

Fig. 2: Dependence of the ionization rate W (in atomic units) of hydrogen atoms in the ground state on the laser power density P (in W/cm^2): dots – our analytical results, solid line – analytical results by Kulyagin and Taranukhin [8]

Figure 3, shows the comparison of our analytical results for W(F) with quantum fully-numerical simulations by Farrelly and Reinhardt [13] (who extended to higher fields the earlier fully-numerical simulations by Hehenberger, McIntosh, and Brändas [15]). It is seen that our analytical results are in a good agreement with the fully-numerical calculation from [13].

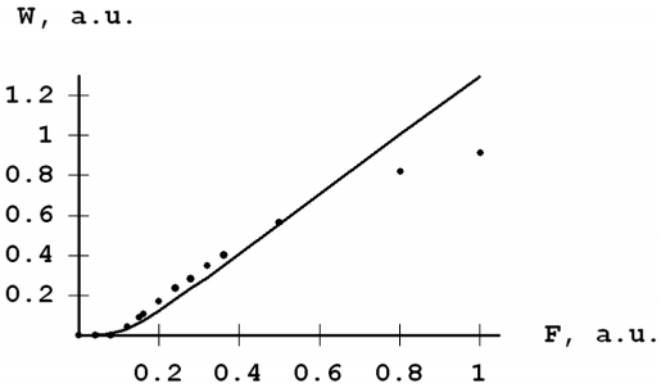

Fig. 3: Dependence of the ionization rate W of hydrogen atoms in the ground state on the laser field F (W and F are in atomic units): dots – our analytical results, solid line – fully – numerical simulations by farrelly and reinhardt [13]

5. CONCLUSIONS

We presented a further development of the analytical model for the tunneling ionization of atoms by a low-frequency laser field of an arbitrary strength, including the strong-field region. The model uses a very accurate approximation of the true ionization barrier/potential in the Schrödinger equation by the effective parabolic barrier/potential – based on the algorithm suggested by Miller and Good [7] and later employed by KT [8] for calculating the ionization rate $W(F)$ of hydrogen atoms from the ground state. We have pointed out and eliminated a number of principal errors made by KT and calculated $W(F)$ much more accurately.

Our results for $W(F)$ differ very significantly from the KT-results. Namely, the dependence of the ionization rate on the laser field is actually monotonic and does not show any effect of the stabilization ("local ionization suppression") claimed by KT. At the same time, our results for $W(F)$ are in a good agreement with the results of quantum fully-numerical calculations [13].

Thus, from the theoretical point of view, the question of a possible stabilization of hydrogen atoms in the ground state with respect to the tunneling ionization has been addressed by us and finally resolved in favor of the absence of such stabilization. Now it might be a good idea to conduct the corresponding experiment to verify this theoretical conclusion.

The characteristic feature of the analytical method (which has been developed and extended here), is not only its accuracy, but also its simplicity. This approach can therefore be extended without difficulty to tunneling ionization for a number of other quantum systems.

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APPENDIX A
CORRECT EXPRESSION FOR $\alpha(F)$ CONTROLLING THE TRANSMISSION COEFFICIENT $D(F)$ IN EQ. (18) AND IN FORMULAS FROM KT-PAPER [8]

The quantity $\alpha(F)$ is incorrectly given by KT. Indeed, in [9], in problem 4 after paragraph 50, the quantity ε (which was re-named to α by KT) is given in atomic units as:

$$\varepsilon = E/k^{1/2}, \qquad \text{... (A.1)}$$

where E is the energy counted from the top of the parabolic barrier (for energies below the top of the barrier, $\varepsilon = -|E|/k^{1/2}$). In other words, in notations of KT, this should be:

$$\alpha = -[U(E)/4 - U_{0p}]/k^{1/2}. \qquad \text{... (A.2)}$$

The quantity k, correctly calculated by KT, is:

$$k = (8/\Delta\eta^2)\,[U(E)/4 - U_{0p}].$$
$$\Delta\eta = \eta_2 - \eta_1. \qquad \text{... (A.3)}$$

After substituting (A.3) in (A.2), we get:

$$\alpha = \Delta\eta\,[U(E)/4 - U_{0p}]/\{8\,[U(E)/4 - U_{0p}]\}^{1/2}$$
$$= -\Delta\eta\,\{[U(E)/4 - U_{0p}]/8\}^{1/2}, \qquad \text{... (A.4)}$$

what differs significantly from the incorrect KT-formula for α.

APPENDIX B
WEAK FIELD EXPANSION OF THE INTEGRAL FROM EQ. (15)

B.1. First we find the weak field limit of the roots $b(F)$, $a(F)$, and $-c(F)$ of the cubic equation:

$$\eta^3 - 2U\eta^2/F + 4\beta_2\,\eta/F + 1/F = 0. \qquad \text{... (B.1)}$$

From a simple analysis it follows that in this limit, $b \to \infty$, while a and c remain finite. Therefore, the asymptotic value of b can be found as the largest root of the truncated equation

$$\eta^3 - 2U\eta^2/F + 4\beta_2\,\eta/F = 0. \qquad \text{... (B.2)}$$

Thus we obtain the first two terms of the asymptotic expansion of b in the form

$$b(F) = (2U/F)\,(1 - \beta^2 F/U^2). \qquad \text{... (B.3)}$$

The asymptotic values of the roots $a(F)$ and $-c(F)$ can be found from another truncated equation obtained by omitting the term η^3 in the cubic equation:

$$-2U\eta^2/F + 4\beta_2\,\eta/F + 1/F = 0. \qquad \text{... (B.4)}$$

As a result, we get:

$$a(F) = [(\beta_2^2 + U/2)^{1/2} + \beta_2]/U,$$
$$c(F) = [(\beta_2^2 + U/2)^{1/2} - \beta_2]/U. \qquad \text{... (B.5)}$$

B.2. The arguments of the elliptic integrals depend on the quantities $d = [(a + c)/(b + c)]^{1/2}$ and $j = [a/(b + c)]^{1/2}$. From the previous section it follows that in the weak field limit we have: $d \to 0$ and $j \to 0$. Thus, the task is to find the expansion of the elliptic integrals as $d \to 0$ and $j \to 0$.

The bracket containing the elliptic integrals of the first kind can be expressed as follows:

$$[\mathbf{K}(1/d^2) - \mathbf{F}(\arcsin d, 1/d^2)]$$
$$= -i\,d\,\mathbf{K}(1 - d^2). \qquad \text{... (B.6)}$$

Let us temporarily denote the argument of the **K**-integral in the right side as k: $k = 1 - d^2$. By introducing $k' = (1 - k^2)$ and noting that in the limit $d \to 0$ it can be approximated as $k' = 2^{1/2}d << 1$, we can use the asymptotic result $\mathbf{K}(k') = \ln(4/k')$ and obtain:

$$[\mathbf{K}(1/d^2) - \mathbf{F}(\arcsin d, 1/d^2)]_{as}$$
$$= -i\,d\ln(2^{3/2}/d). \qquad \text{... (B.7)}$$

The bracket containing the elliptic integrals of the second kind can be represented in the form:

$$[\mathbf{E}(1/d^2) - \mathbf{E}(\arcsin d, 1/d^2)]$$
$$= (i/d)\,I_1(d),$$

$$I_1(d) = \int_d^1 dt\left[\left(t^2 - d^2\right)/\left(1 - t^2\right)\right]^{1/2} \qquad \text{... (B.8)}$$

By expanding $I_1(d)$ in the Taylor series, we obtain:

$$I_1(d) = I_1(0) + d\,I_1'(0) = 1 - d^2\,I_2(d),$$
$$I_2(d) = dt\,[(t^2 - d^2)\,(1 - t^2)]^{-1/2} \qquad \text{... (B.9)}$$

(here $I_1'(z)$ is the derivative of $I_1(z)$). For obtaining the asymptotic value of $I_2(d)$, we note that:

$$[\mathbf{K}(1/d^2) - \mathbf{F}(\arcsin d, 1/d^2)]$$
$$= -i\,d\,I_2(d). \qquad \text{... (B.10)}$$

By comparing (B.10) and (B.6) we find:

$$I_2(d) = \mathbf{K}(1 - d^2). \qquad \text{... (B.11)}$$

Then by using the asymptotic value of **K** $(1 - d^2)$ from (B.7), we get:

$$[\mathbf{E}\,(1/d^2) - \mathbf{E}\,(\arcsin d,\, 1/d^2)]_{as} = i\,[1/d - d\,\ln(2^{3/2}/d)]. \quad \text{... (B.12)}$$

The bracket containing the elliptic integrals of the third kind can be represented in the form:

$$[\Pi\,(1/(d^2 - j^2),\, 1/d^2) - \Pi\,(1/(d^2 - j^2),\, \arcsin d,\, 1/d^2)] = i\,d\,(d^2 - j^2) \int_d^1 dt \left(t^2 - d^2 + j^2\right)^{-1} \left[\left(t^2 - d^2\right)\left(1 - t^2\right)\right]^{-1/2} \quad \text{... (B.13)}$$

The integrand in (B.13) has singularities at $t = d$ and at $t = 1$, where it goes to positive infinity. Inside the integration range, the integrand has a minimum at $t = 3^{1/2}/2$. The location of the minimum as well as the subsequent results are obtained in the limit $d << 1$ (we remind that $j < d$).

We divide the integration range in two parts: from d to $d^{P(n)}$, the corresponding integral being denoted I_{31}, and from $d^{P(n)}$ to 1, the corresponding integral being denoted I_{32}. Here $P(n) = 1/2^n$, where $n = 1, 2, 3, \ldots$, so that $d < d^{P(n)} < 1$. The integral I_{31} can be approximated as follows:

$$I_{31} = \int_d^{P(n)} dt\; j^{-2} \left(t^2 - d^2\right)^{-1/2}. \quad \text{... (B.14)}$$

After integrating I_{31} from (B.14) analytically and using $n >> 1$, we obtain:

$$I_{31} = j^{-2} \ln (2/d). \quad \text{... (B.15)}$$

The integral I_{32} can be approximated as follows:

$$I_{32} = \int_{P(n)}^1 dt\; t^{-3} \left(1 - t^2\right)^{-1/2}. \quad \text{... (B.16)}$$

After integrating I_{32} from (B.16) analytically and using $n >> 1$, we find that $I_{32} = \text{const} \sim 1$, so that it can be disregarded compared to $I_{31} = j^{-2} \ln (2/d)$ since $j < d << 1$.

Thus we get the asymptotic expression for the right side of (B.13) is:

$$[\Pi\,(1/(d^2 - j^2),\, 1/d^2) - \Pi\,(1/(d^2 - j^2),\, \arcsin d,\, 1/d^2)]_{as} = i\,d\,(d^2 - j^2)\, j^{-2} \ln (2/d). \quad \text{... (B.17)}$$

By substituting (B.7)), (B.12), and (B.17) in (4) we obtain our Eq. (22) for $\text{Int}_{as}(F)$.

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