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Superdiffusive Transport of Biberman-Holstein Type for a Finite Velocity of Carriers: General Solution and the Problem of Automodel Solutions

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ABSTRACT: The present work is aimed at generalizing the method of approximate automodel solutions for the Green's function of the superdiffusive (nonlocal) transport equations (J. Phys. A: Math. Theor. 49 (2016) 255002) onto the case of a finite velocity of carriers. This problem covers the case of resonant photon's transport in astrophysical gases and plasmas. We treat a model case, namely, one-dimensional transport with a model, power-law step-length probability distribution function (PDF). This is close to the Biberman-Holstein transport for the line shapes of Doppler and/or Lorentz type because the respective line shape's wings produce the power-law asymptotic of the step-length PDF at large distances (i.e. for the distant flights of the resonant photon). Here, general solution for the Green's function is found, and an analytic result for the asymptotic far ahead of the perturbation front is derived.

1. INTRODUCTION

The analysis of the Green's function of the non-stationary Biberman-Holstein equation [1-4] for radiative transfer in plasmas and gases has shown [5, 6] that there is an approximate automodel solution based on three scaling laws: for the propagation front (i.e. relevant-to-superdiffusion average displacement of perturbation's carrier from an instant point source) and asymptotic solutions far beyond and far ahead of the propagation front. All these scaling laws are determined essentially by the long-free-path carriers (named, by Mandelbrot [7], Lévy flights, cf. e.g. [8]). The validity of the suggested automodel solution was proved by its comparison with analytical solutions by Veklenko [9] in the 3D case of the Biberman–Holstein equation of the resonance radiation transfer for various (Doppler, Lorentz, Voigt and Holtsmark) spectral line shapes with complete redistribution over frequency (within spectral line width) in the elementary act of the resonance scattering (i.e. absorption and subsequent emission) of the photon by an atom/ion. Scaling laws of Biberman-Holstein equation Green's function and the implications for algorithms of numerical modeling of superdiffusive transport are considered in [10]. The approach [5] was extended in [6] on a wide class of non-stationary superdiffusive transport on a uniform background with a simple long-tailed step-length probability distribution function (PDF) with various power-law exponents. The results of accuracy analysis of automodel solutions for Lévy flight-based transport, including the resonance radiative transfer and a simple general model, is reported in [11].

The present work is aimed at generalizing the method of approximate automodel solutions [6] onto the case of a finite velocity of carriers. This problem covers the case of resonant photon's transport in astrophysical gases and plasmas (cf. [12, 13]). The general features of superdiffusion, based on Lévy flights, are recognized and applied in many fields (see, e.g., popular article [14]).

Firstly, we present accurate derivation of the one-dimensional (1D) Biberman-Holstein equation with allowance for the finite velocity of the photons and formulate the 1D transport equation with a model, power-law step-length probability distribution function (PDF), similarly to the model [6], and a finite velocity of carriers (sec. 2). Note that such a PDF describes the transport which is close to the Biberman-Holstein transport for the line shapes of Doppler and/or Lorentz type because the respective line shape's wings produce the power-law asymptotic of the step-length PDF at large distances (see, e.g., the survey [4]). The general solution for the Green's function of the Biberman-Holstein type equation is derived in Sec. 3. A transition, in the general solution, to the limiting case of the infinite speed of light, which is applicable to laboratory plasmas and gases, is considered in Sec. 4. The problem of scaling laws and approximate automodel solution [6], generalized to the case of a finite velocity of carriers, is treated in Sec. 5, and an analytic result for the asymptotic far ahead of the perturbation front is derived in Sec. 6.

2. FORMULATION OF THE PROBLEM

We consider a one-dimensional (1D) problem of perturbation transport by Lévy flights of the finite propagation velocity «c» of the carriers. First, we present accurate derivation of the 1D Biberman-Holstein equation with allowance for the finite velocity of the photons, which is helpful for the deriving the general solution of the problem.

The equation for the Green's function of the density $f(x, t)$ of excited two-level atoms (ions) in the Biberman-Holstein approach [1, 2] has the form:

$$\frac{\partial f(x,t)}{\partial t} = -\left(\frac{1}{\tau}+\sigma\right) f(x,t) + \int_{-\infty}^{+\infty} \kappa(\omega,x)\left[I_+(\omega,x,t)+I_-(\omega,x,t)\right] d\omega + \delta(x)\delta(t), \tag{1}$$

where τ is the lifetime of the excited atomic state with respect to spontaneous radiative decay, σ is the rate of (collisional) quenching of excitation; the last term is the instant point source of excited atoms, different from population by the absorption of the resonant photon (e.g., collisional excitation), $\kappa(\omega, x)$ is the absorption coefficient at photon's frequency ω at the point x, $I_+(\omega, x, t)$ и $I_-(\omega, x, t)$ are the radiation intensities at frequency ω, coming to the point x from the left and the right, respectively.

The balance equations for radiation intensities at frequency ω, coming to the point x from the left and the right, respectively, have the form:

$$\frac{1}{c}\frac{\partial I_+(\omega,x,t)}{\partial t} + \frac{\partial I_+(\omega,x,t)}{\partial x} = \frac{1}{2\tau} f(x,t)e(\omega) - \kappa(\omega,x)I_+(\omega,x,t), \tag{2}$$

$$\frac{1}{c}\frac{\partial I_-(\omega,x,t)}{\partial t} - \frac{\partial I_-(\omega,x,t)}{\partial x} = \frac{1}{2\tau} f(x,t)e(\omega) - \kappa(\omega,x)I_-(\omega,x,t), \tag{3}$$

where $e(\omega)$ and $a(\omega)$ are the normalized spectral distributions, the line shapes, of the carriers for emission and absorption processes, respectively:

$$\int_{-\infty}^{+\infty} e(\omega)d\omega = \int_{-\infty}^{+\infty} a(\omega)d\omega = 1. \tag{4}$$

Initial and boundary conditions for Eqs. (2) and (3) are as follows:

$$\begin{aligned} &I_+(\omega,x,0) = I_-(\omega,x,0) = 0, \\ &I_+(\omega,x\to-\infty,t) = I_-(\omega,x\to\infty,t) = 0. \end{aligned} \tag{5}$$

The solutions of Eqs. (2) and (3), which obey these conditions, have the following form:

$$I_{+}(\omega,x,t)=\frac{e(\omega)}{2\tau}\int_{-\infty}^{x}dx'\,f(x',t-\frac{|x-x'|}{c})\theta(t-\frac{|x-x'|}{c})\exp\left(-\int_{x'}^{x}\kappa(\omega,x'')dx''\right),\tag{6}$$

$$I_{-}(\omega,x,t)=\frac{e(\omega)}{2\tau}\int_{x}^{+\infty}dx'\,f(x',t-\frac{|x-x'|}{c})\theta(t-\frac{|x-x'|}{c})\exp\left(-\int_{x}^{x'}\kappa(\omega,x'')dx''\right).\tag{7}$$

While substituting the solutions (6) and (7) into equation (1), let us consider separately the term with the integral over frequency:

$$\begin{aligned}&\int_{-\infty}^{+\infty}\kappa(\omega,x)\left[I_{+}(\omega,x,t)+I_{-}(\omega,x,t)\right]d\omega=\frac{1}{2\tau}\int_{-\infty}^{+\infty}d\omega\kappa(\omega,x)e(\omega)\times\\&\times\int_{-\infty}^{+\infty}dx'\,f(x',t-\frac{|x-x'|}{c})\theta(t-\frac{|x-x'|}{c})\exp\left(-\left|\int_{x'}^{x}\kappa(\omega,x'')dx''\right|\right).\end{aligned}\tag{8}$$

In a uniform medium, the absorption coefficient depends only on the absorbed photon's frequency, the integral in the exponent in Eq. (8) becomes simpler. The integral over frequency may be expressed in terms of the probability density $W(\rho)$, which is a step-length probability distribution function (PDF) with respect to the process of photon's emission and subsequent absorption after passing the distance ρ:

$$\begin{gathered}T(\rho)=\int_{-\infty}^{+\infty}e(\omega)\exp\left(-\kappa(\omega)\rho\right)d\omega,\qquad W(\rho)=\frac{1}{2}\int_{-\infty}^{+\infty}\kappa(\omega)e(\omega)\exp\left(-\kappa(\omega)\rho\right)d\omega.\\W(\rho)=-\frac{dT(\rho)}{2d\rho},\qquad \rho=|x|.\end{gathered}\tag{9}$$

The right part of Eq. (8) becomes

$$\frac{1}{\tau}\int_{-\infty}^{+\infty}dx'\,f(x',t-\frac{|x-x'|}{c})\theta(t-\frac{|x-x'|}{c})W(|x-x'|).\tag{10}$$

Thus, we arrive at the following equation for the excited atom density, which is a one-dimensional extension of the Biberman-Holstein equation to the case of a finite velocity «c» of the energy carriers (cf. [12, 13]):

$$\frac{\partial f(x,t)}{\partial t}=-\left(\frac{1}{\tau}+\sigma\right)f(x,t)+\frac{1}{\tau}\int_{-\infty}^{+\infty}dx'W(|x-x'|)f(x',t-\frac{|x-x'|}{c})\theta(t-\frac{|x-x'|}{c})+\delta(x)\delta(t).\tag{11}$$

It appears, however, that for deriving the general solution of the problem it is worth to come back to equations (1)-(3) and try to solve these with the Fourier and Laplace transformations. Equation (11) is very helpful for testing the asymptotic of the Green's function far ahead of the perturbation front, obtained from general solution in Sec. 6.

3. GENERAL SOLUTION FOR GREEN'S FUNCTION

We proceed with transition to dimensionless variables x and t, normalized to κ_0 (absorption coefficient in the center of the line shape) and τ, respectively. This gives the absorption coefficient $\kappa(\omega)=\frac{\kappa_0}{a_0}a(\omega)$. The Fourier transforms of the functions $f(x, t)$, $I_+(\omega, x, t)$ and $I_-(\omega, x, t)$ in the dimensionless equations (1)-(3) are defined as follows:

$$f(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\widehat{f}(p,t)e^{ipx}dp, \qquad I_{\pm}(\omega,x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\widehat{I}_{\pm}(\omega,p,t)e^{ipx}dp. \tag{12}$$

Using the boundary condition, Eq. (5), we derive a set of equations for the Fourier transforms:

$$\frac{\partial\widehat{f}(p,t)}{\partial t}=-\left(1+\sigma\tau\right)\widehat{f}(p,t)+\frac{\kappa_0\tau}{a_0}\int_{-\infty}^{+\infty}a(\omega)\left[\widehat{I}_{+}(\omega,p,t)+\widehat{I}_{-}(\omega,p,t)\right]d\omega+\frac{1}{\sqrt{2\pi}}\kappa_0\delta(t), \tag{13}$$

$$\frac{1}{c\tau}\frac{\partial\hat{I}_{+}(\omega,p,t)}{\partial t}+ip\kappa_0\hat{I}_{+}(\omega,p,t)=\frac{1}{2\tau}\widehat{f}(p,t)e(\omega)-\frac{\kappa_0}{a_0}a(\omega)\hat{I}_{+}(\omega,p,t), \tag{14}$$

$$\frac{1}{c\tau}\frac{\partial\hat{I}_{-}(\omega,p,t)}{\partial t}-ip\kappa_0\hat{I}_{-}(\omega,p,t)=\frac{1}{2\tau}\widehat{f}(p,t)e(\omega)-\frac{\kappa_0}{a_0}a(\omega)\hat{I}_{-}(\omega,p,t), \tag{15}$$

with the initial condition

$$\widehat{I}_{+}(\omega,p,0)=\widehat{I}_{-}(\omega,p,0)=0. \tag{16}$$

The solution of the ordinary differential equation (14) with initial condition (16) is as follows:

$$\hat{I}_{+}(\omega,p,t)=\frac{c}{2}e(\omega)\int_0^t\exp\left[-c\tau\kappa_0\left(ip+\frac{a(\omega)}{a_0}\right)(t-t')\right]\widehat{f}(p,t')dt', \tag{17}$$

and the solution of Eq. (15) with the same condition (16) —

$$\hat{I}_{-}(\omega,p,t)=\frac{c}{2}e(\omega)\int_0^t\exp\left[-c\tau\kappa_0\left(-ip+\frac{a(\omega)}{a_0}\right)(t-t')\right]\widehat{f}(p,t')dt'. \tag{18}$$

The latter solution can also be obtained by the replacement $\kappa_0\to-\kappa_0, a_0\to-a_0$ in Eq. (17).

Substitution of Eqs. (17) and (18) into Eq. (13) and the change of the integration order yields:

$$\begin{aligned}\frac{\partial\widehat{f}(p,t)}{\partial t}&=-\left(1+\sigma\tau\right)\widehat{f}(p,t)+\frac{\kappa_0}{\sqrt{2\pi}}\delta(t)+\\&+c\tau\kappa_0\int_0^t dt'\widehat{f}(p,t')\cos\left[c\tau\kappa_0p(t-t')\right]\int_{-\infty}^{+\infty}\frac{a(\omega)}{a_0}e(\omega)\exp\left(-\frac{a(\omega)}{a_0}c\tau\kappa_0(t-t')\right)d\omega.\end{aligned} \tag{19}$$

The internal integral in Eq. (19) may be expressed in terms of the PDF (9) in the dimensionless variables. Correspondingly, equation (19) may be rewritten in the form

$$\frac{\partial\widehat{f}(p,t)}{\partial t}=-\left(1+\sigma\tau\right)\widehat{f}(p,t)+\frac{\kappa_0}{\sqrt{2\pi}}\delta(t)+2c\tau\kappa_0\int_0^t\widehat{f}(p,t')\cos\left[c\tau\kappa_0p(t-t')\right]W\left(c\tau\kappa_0(t-t')\right)dt'. \tag{20}$$

We will consider the power-law PDF:

$$T(\rho) = \frac{1}{(1+\rho)^{\gamma}}, \qquad W(\rho) = \frac{\gamma}{2(1+\rho)^{\gamma+1}}, \qquad 0 < \gamma < 2, \qquad \rho = |x|. \tag{21}$$

Introducing the notation

$$k(p,t) \equiv 2c\tau\kappa_0 \cos(c\tau\kappa_0 pt) W(c\tau\kappa_0 t) = \frac{\gamma c\tau\kappa_0 \cos(c\tau\kappa_0 pt)}{(1+c\tau\kappa_0 t)^{\gamma+1}}. \tag{22}$$

we get Eq. (20) in a more compact form:

$$\frac{\partial \widehat{f}(p,t)}{\partial t} = -\left(1+\sigma\tau\right)\widehat{f}(p,t) + \frac{\kappa_0}{\sqrt{2\pi}}\delta(t) + \int_0^t k(p,t-t')\widehat{f}(p,t')dt'. \tag{23}$$

The initial condition $\widehat{f}(p,t=0) = \dfrac{\kappa_0}{\sqrt{2\pi}}$ is taken into account here.

To solve the equation (23), we use the Laplace transform:

$$\begin{aligned} F(p,s) &= \int_0^{+\infty} e^{-st}\widehat{f}(p,t)dt, & K(p,s) &= \int_0^{+\infty} e^{-st}k(p,t)dt, \\ \widehat{f}(p,t) &= \frac{1}{2\pi i}\lim_{\alpha\to+0}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}F(p,s)ds, & k(p,t) &= \frac{1}{2\pi i}\lim_{\alpha\to+0}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}K(p,s)ds. \end{aligned} \tag{24}$$

This gives the following equation:

$$sF(p,s) = -\left(1+\sigma\tau\right)F(p,s) + \frac{\kappa_0}{\sqrt{2\pi}} + K(p,s)F(p,s), \tag{25}$$

where

$$K(p,s) = \int_0^{+\infty} e^{-st}k(p,t)dt = \gamma c\tau\kappa_0 \int_0^{+\infty} e^{-st}\frac{\cos(c\tau\kappa_0 pt)}{(1+c\tau\kappa_0 t)^{\gamma+1}}dt = \gamma\int_0^{+\infty}\frac{e^{-su/c\tau\kappa_0}\cos(pu)}{(1+u)^{\gamma+1}}du. \tag{26}$$

For the Laplace transform of the density, we obtain

$$F(p,s) = \frac{\kappa_0}{\sqrt{2\pi}}\frac{1}{s+1+\sigma\tau-\gamma\displaystyle\int_0^{+\infty}\frac{e^{-su/c\tau\kappa_0}\cos(pu)}{(1+u)^{\gamma+1}}du}. \tag{27}$$

and, finally, for the density,

$$f(x,t) = \frac{\kappa_0}{(2\pi)^2 i}\int_{-\infty}^{+\infty} dp e^{ipx}\lim_{\alpha\to+0}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{e^{st}ds}{s+1+\sigma\tau-\gamma\displaystyle\int_0^{+\infty}\frac{e^{-su/c\tau\kappa_0}\cos(pu)}{(1+u)^{\gamma+1}}du}. \tag{28}$$

In what follows, we omit the quenching, σ = 0, because, in a homogeneous medium, this effects is described merely by an exponential decay in time. Introducing the new variable, $s=\alpha+iy$, $\alpha=+0$, after some transformations we obtain general solution for the Green's function:

$$f(x,t)=\frac{\kappa_0}{(2\pi)^2}\int_{-\infty}^{+\infty}dp\cos(px)\times$$
$$\times\left[\int_{-\infty}^{+\infty}\frac{\cos(yt)\left\{1-\gamma\int_0^{+\infty}\frac{\cos\left(yu/c\tau\kappa\right)\cos(pu)}{(1+u)^{\gamma+1}}du\right\}+\sin(yt)\left\{y+\gamma\int_0^{+\infty}\frac{\sin\left(yu/c\tau\kappa\right)\cos(pu)}{(1+u)^{\gamma+1}}du\right\}}{\left[1-\gamma\int_0^{+\infty}\frac{\cos\left(yu/c\tau\kappa\right)\cos(pu)}{(1+u)^{\gamma+1}}du\right]^2+\left[y+\gamma\int_0^{+\infty}\frac{\sin\left(yu/c\tau\kappa\right)\cos(pu)}{(1+u)^{\gamma+1}}du\right]^2}dy\right]. \tag{29}$$

4. THE LIMIT OF INFINITE VELOCITY OF CARRIERS

Let's consider the limiting case of Eq. (29), where the velocity of carriers is taken infinite:

$$\lim_{c\to+\infty}f(x,t)=\frac{\kappa_0}{(2\pi)^2}\int_{-\infty}^{+\infty}dp\cos(px)\left[\int_{-\infty}^{+\infty}\frac{\left(1-\gamma\int_0^{+\infty}\frac{\cos(pu)}{(1+u)^{\gamma+1}}du\right)\cos(yt)+y\sin(yt)}{\left[1-\gamma\int_0^{+\infty}\frac{\cos(pu)}{(1+u)^{\gamma+1}}du\right]^2+y^2}dy\right]. \tag{30}$$

Introducing $\beta\equiv1-\gamma\int_0^{+\infty}\frac{\cos(pu)}{(1+u)^{\gamma+1}}du$, we get

$$\lim_{c\to+\infty}f(x,t)=\frac{\kappa_0}{(2\pi)^2}\int_{-\infty}^{+\infty}dp\cos(px)\left[\beta\int_{-\infty}^{+\infty}\frac{\cos(yt)}{\beta^2+y^2}dy+\int_{-\infty}^{+\infty}\frac{y\sin(yt)}{\beta^2+y^2}dy\right]. \tag{31}$$

The integrals may be calculated analytically, and we finally obtain the result,

$$\lim_{c\to+\infty}f(x,t)=\frac{\kappa_0}{2\pi}\int_{-\infty}^{+\infty}dp\cos(px)\exp\left[-tp\int_0^{+\infty}\frac{\sin(pu)}{(1+u)^{\gamma}}du\right], \tag{32}$$

which precisely coincides with Eq. (18) in [6].

5. ALGORITHM OF VERIFYING THE SELF-SIMILARITY

Let us consider the method of approximate automodel solution, developed in [6], in the view of its extension to the case of a finite velocity of carriers. We suggest the following asymptotic expression for the Green's function far ahead of the perturbation front, i.e. at $\rho>>\rho_{fr}(t)>>1$ (or, equivalently $1<<t<<t_{fr}(\rho)$), where $\rho=|x|$ and $\rho_{fr}(t)$ is the propagation front (cf. Eq. (6) in [6]):

$$f(x,t) = \left(t - \frac{|x|}{c}\right) W(|x|)\theta(t - \frac{|x|}{c}). \tag{33}$$

The front propagation law, $\rho_{fr}(t)$, has to generalize equation (5) in [6]. Far beyond the perturbation front, i.e. at $\rho << \rho_{fr}(t)$ (or, equivalently $t >> t_{fr}(\rho) >> 1$), the asymptotic of the Green's function is supposed to be similar to Eq. (7) in [6]. This means that the solution has the form of a step with the width proportional to the distance from the source to the perturbation front:

$$f(x,t) \sim \frac{1}{2\rho_{fr}(t)} \theta\left(\rho_{fr}(t) - |x|\right), \tag{34}$$

where $\theta(t)$ is the Heaviside function.

The automodel solution may be taken in the form which generalizes Eqs. (9)-(11) in [6]:

$$f(x,t) = \left(t - \frac{\rho}{c}\right) W\left(\rho g\left(\frac{\rho_{fr}}{\rho}\right)\right)\theta(t - \frac{\rho}{c}), \qquad \rho \equiv |x|, \qquad g(s) = \begin{cases} 1, & s << 1, \\ s, & s >> 1. \end{cases} \tag{35}$$

For particular form of the PDF, one has:

$$f(x,t) = \left(t - \frac{\rho}{c}\right) \frac{\gamma}{2\left[1 + \rho g\left(\frac{\rho_{fr}(t)}{\rho}\right)\right]^{\gamma+1}} \theta(t - \frac{\rho}{c}). \tag{36}$$

For $0 < t < \frac{\rho}{c}$, the function g in Eq. (36) is related to the exact solution f_{exact} (29) similarly to the relation of Eq. (25) in [6] to the respective exact solution:

$$Q_W(x,t) = \frac{1}{\rho} \hat{W}^{-1}\left(\frac{f_{exact}(x,t)}{t - \rho/c}\right), \tag{37}$$

where $\hat{W}^{-1}(y) = \left(\frac{\gamma}{2y}\right)^{1/(\gamma+1)} - 1$ is the inverse function to $W(x)$. Rewriting Eq. (37), we get

$$Q_W(x,t) = \frac{1}{\rho}\left[\left(\frac{\gamma(t - \rho/c)}{2 f_{exact}(x,t)}\right)^{\frac{1}{\gamma+1}} - 1\right], \qquad \rho \equiv |x|. \tag{38}$$

Notice, that Eqs. (26) and (27) in [6] define the phenomenon of approximate automodel solution (i.e. approximate self-similarity):

$$Q_W\left(\rho, t(\rho,s)\right) = Q_{W1}(s,\rho) = g(s),$$
$$Q_W\left(\rho(t,s), t\right) = Q_{W2}(s,\rho) = g(s), \tag{39}$$

where $s \equiv \dfrac{\rho_{fr}(t)}{\rho}$.

6. ASYMPTOTIC FAR AHEAD OF THE PERTURBATION FRONT

Let us consider Eq. (29) in the limit $t \to 0$, $\rho_{fr} \to \infty$. In this limit, the following functions appear in Eq. (29):

$$I_1 \equiv I_1(y,p) \equiv \int_0^{+\infty} \frac{\cos\left(yu/c\tau\kappa\right)\cos(pu)}{(1+u)^{\gamma+1}}du, \qquad I_2 \equiv I_2(y,p) \equiv \int_0^{+\infty} \frac{\sin\left(yu/c\tau\kappa\right)\cos(pu)}{(1+u)^{\gamma+1}}du.$$

The Green's function takes the form

$$f(x,t) = \frac{\kappa_0}{(2\pi)^2}\int_{-\infty}^{+\infty} dp\cos(px)\left[\int_{-\infty}^{+\infty} \frac{\cos(yt)\left\{1-\gamma I_1\right\} + \sin(yt)\left\{y+\gamma I_2\right\}}{\left[1-\gamma I_1\right]^2 + \left[y+\gamma I_2\right]^2}dy\right]. \tag{40}$$

We expand the denominator in the square brackets in the series of a small values of I_1 and I_2, omitting the terms without *cos*(*pu*), because these give zero contribution to the resulted integral. After the change of the integration order, we get:

$$\begin{aligned} f(x,t) = {} & \frac{\kappa_0}{(2\pi)^2}\int_{-\infty}^{+\infty} dp\cos(px)\int_0^{+\infty} du\frac{\cos(pu)}{(1+u)^{\gamma+1}} \times \\ & \times\left\{\begin{aligned} & -\gamma\int_{-\infty}^{+\infty} dy\frac{\cos(yt)}{1+y^2}\cos\left(\frac{yu}{c\tau\kappa}\right) + \gamma\int_{-\infty}^{+\infty} dy\frac{\sin(yt)}{1+y^2}\sin\left(\frac{yu}{c\tau\kappa}\right) + \\ & +2\gamma\int_{-\infty}^{+\infty} dy\frac{\cos(yt)}{(1+y^2)^2}\cos\left(\frac{yu}{c\tau\kappa}\right) + 2\gamma\int_{-\infty}^{+\infty} dy\frac{y\sin(yt)}{(1+y^2)^2}\cos\left(\frac{yu}{c\tau\kappa}\right) - \\ & -2\gamma\int_{-\infty}^{+\infty} dy\frac{y\cos(yt)}{(1+y^2)^2}\sin\left(\frac{yu}{c\tau\kappa}\right) - 2\gamma\int_{-\infty}^{+\infty} dy\frac{y^2\sin(yt)}{(1+y^2)^2}\sin\left(\frac{yu}{c\tau\kappa}\right) \end{aligned}\right\}. \end{aligned} \tag{41}$$

The analytic calculation of the integrals over y in Eq. (41) yields:

$$f(x,t) = \frac{\kappa_0}{(2\pi)^2}\int_{-\infty}^{+\infty} dp\cos(px)\int_0^{+\infty} du\frac{\cos(pu)}{(1+u)^{\gamma+1}}\pi\gamma\left\{\left[\left|t-\frac{u}{c\tau\kappa}\right| + \left(t-\frac{u}{c\tau\kappa}\right)\right]e^{-\left|t-\frac{u}{c\tau\kappa}\right|} + \left(t+\frac{u}{c\tau\kappa}\right)e^{-\left(t+\frac{u}{c\tau\kappa}\right)}\right\}. \tag{42}$$

The integral over p may be calculated analytically, that yields

$$f(x,t)=\frac{\kappa_0}{2}\gamma\int_0^{c\tau\kappa t}\frac{du}{(1+u)^{\gamma+1}}\{\delta(u-x)+\delta(u+x)\}\left[t-\frac{u}{c\tau\kappa}\right]e^{-\left(t-\frac{u}{c\tau\kappa}\right)}+$$
$$+\frac{\kappa_0}{4}\gamma\int_0^{+\infty}\frac{du}{(1+u)^{\gamma+1}}\{\delta(u-x)+\delta(u+x)\}\left(t+\frac{u}{c\tau\kappa}\right)e^{-\left(t+\frac{u}{c\tau\kappa}\right)}. \tag{43}$$

Integrating over u and taking into account Eq. (21), we get

$$f(x,t)=\kappa_0\left(t-\frac{|x|}{c\tau\kappa}\right)W(|x|)\theta(t-\frac{|x|}{c\tau\kappa})e^{-\left(t-\frac{|x|}{c\tau\kappa}\right)}+\frac{\kappa_0}{2}\left(t+\frac{|x|}{c\tau\kappa}\right)W(|x|)e^{-\left(t+\frac{|x|}{c\tau\kappa}\right)}. \tag{44}$$

In the case of limit under consideration, the second term in Eq. (44) tends to zero, and the exponent tends to unity. Coming back to dimensional variables, for the asymptotic far ahead of the perturbation front, we finally obtain:

$$f(x,t)=\frac{1}{\tau}\left(t-\frac{|x|}{c}\right)W(|x|)\theta(t-\frac{|x|}{c}). \tag{45}$$

This expression, being valid for the infinite velocity of carriers, $c\to\infty$, precisely coincides with Eq. (6) in [6].

Note, that Eq. (45) may also be derived from Eq. (11), if one (i) neglects the spread of the initial perturbation around the point of the source, (ii) looks on the density of excited atoms very far from the initial source, and (iii) neglects the exchange of the neighboring atoms with the excitation. Thus, the validity of Eq. (45) is proved in two quite different ways.

7. CONCLUSIONS

The results of analyzing the general solution derived for the one-dimensional (1D) superdiffusive transport equation of the Biberman-Holstein type equation, with a power-law step-length probability distribution function (PDF) and with allowance for a finite velocity of carriers (photons), allow to draw the following conclusions.

The general solution is tested to give the correct transition to the limiting case of the infinite speed of light, which is applicable to laboratory plasmas and gases.

An analytic result, which is derived from the general solution for the asymptotic far ahead of the perturbation front, gives the scaling law which, for the infinite velocity of carriers, coincides with the respective results for the exact solution and approximate automodel solution, derived in [6].

These results justify the formulation of a rather simple extension of the approach [6] to the case of a finite velocity of carriers. The suggested extension will be verified in the future work.

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